Topology Notes
Math 131 — Harvard University
Spring 2001

1. **Countable metric spaces.** Theorem. Every countable metric space X is totally disconnected.

   Proof. Given $x \in X$, the set $D = \{d(x,y) \ : \ y \in X\}$ is countable; thus there exist $r_n \to 0$ with $r_n \not\in D$. Then $B(x, r_n)$ is both open and closed, since the sphere of radius r_n about x is empty. Thus the largest connected set containg x is x itself.

2. **A countable connected Hausdorff space.** (Due to Urysohn?) Let H be the set of points in the closed upper half-plane with rational coordinates. Any point in H is the vertex of a (possibly degenerate) equilateral triangle $T(p)$ resting on the x-axis. Let a neighborhood of p consist of p itself union a pair of intervals on the x-axis about the feet of $T(p)$. Since a line with slope 60° passes through at most one rational point, the feet of $T(p)$ and $T(q)$ are disjoint if $p \neq q$ and thus H is Hausdorff. On the other hand, any open set contains an interval on the x-axis, and the closure of such an interval consists of two bands forming an infinite copy of the letter V. Any two of these V's intersect, so H is connected.

3. **Free products.** Let G_i be a collection of groups. As sets, we regard all the groups as having the same identity element e; otherwise the groups are disjoint. The *free product* $G = * \, G_i$ is defined as follows.

   First, a *word* w is a finite sequence of elements $w = (a_1, a_2, \ldots, a_n)$ with $a_j \in \bigcup G_i$. We define a product on words by concatenation; that is,

   $$w * v = (a_1, \ldots, a_n) * (b_1, \ldots, b_m) = (a_1, \ldots, a_n, b_1, \ldots, b_m).$$

   A word is *reduced* if a_j and a_{j+1} belong to different G_i's for all j. As a set, G is the collection of all reduced words.

   Second, given a word w that may not be reduced, a *shortening* of w is any word of the form

   $$w' = (a_1, \ldots, a_j a_{j+1}, \ldots, a_n),$$

   where a_j and a_{j+1} lie in the same group G_i. By iterated shortening we eventually arrive at a reduced word.

Theorem. All shortenings lead to the same reduced word $r(w)$.

Proof. The proof is by induction on the length of w, being clear if the length of w is 1. Now let w' and w'' be two different shortenings of w, working on the letters with indices $j < k$. In other words, w' is obtained by replacing (a_j, a_{j+1}) with $(a_j a_{j+1})$, and w'' is obtained by replacing (a_k, a_{k+1}) with $(a_k a_{k+1})$.

If the letters are far enough apart, we can then simplify the k-pair in w' and the j-pair in w'' to obtain a common simplification w''' of both. Thus there exists a chain of simplifications of w' and w'' leading to the same reduced word. By induction, all simplification of w' and w'' lead to the same word, so we are done.

It may however happen that $k = j + 1$. But then we have a triple of letters (a_j, a_{j+1}, a_{j+2}) all lying in the same group G_i. Thus we can simplify w' and w'' to obtain a word w''' where this triple of letters is replaced by $a_j a_{j+1} a_{j+2}$. Again we are done by induction. ■

Cor. We have $r(v * w) = r(r(v) * w) = r(v * r(w)) = r(r(v) * r(w))$.

We now define the product operation on G by $vw = r(v * w)$. Because of the preceding theorem, it is now easy to verify the group axioms. For example, the product is associative because

$$(uv)w = r(r(u * v) * w) = r(u * v * w) = r(u * r(v * w)) = u(vw).$$

Theorem. Any collection of homomorphisms $\phi_i : G_i \to H$ extends to a unique homomorphism $\phi : G \to H$.

Proof. Define a map $\phi : \bigcup G_i \to H$ by taking the union of the ϕ_i, and extend ϕ to a map on words to H by $\phi(a_1, \ldots, a_n) = \phi(a_1) \cdots \phi(a_n)$. By definition, we have

$$\phi(v * w) = \phi(v)\phi(w).$$

It is easy to check that $\phi(w) = \phi(w')$ whenever w' is a simplification of w. Therefore $\phi(r(w)) = \phi(w)$. Restricting to the set G of reduced words, we find

$$\phi(vw) = \phi(r(v * w)) = \phi(v * w) = \phi(v)\phi(w),$$

so ϕ is a homomorphism. ■