The danger of difference: nonmeasurable sets $A - A$ with $A \subset \mathbb{R}$ measurable

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1 Introduction

A standard homework exercise in real analysis is to show that $A-A$ contains a nonempty interval whenever $A \subset \mathbb{R}$ has positive Lebesgue measure. (For the proof one can use Littlewood's first principle, which states that A is nearly a finite union of intervals.)

Of course A does not need to have positive measure for its difference set to contain an interval; in fact the standard Cantor middle–thirds set $K \subset [0,1]$ has measure zero, and $K - K = [-1,1]$.

What is less well–known is that the standard dictum that 'reasonable operations preserve measurability' fails here: even if A is measurable, its difference set $A - A$ need not be. The Cantor set example already hints that such a pathology might occur: any subset of K is measurable, but $K - K$ contains many nonmeasurable sets, so one might guess that there is an $A \subset K$ such that $A - A$ is nonmeasurable. We will show this is indeed the case.

The proof will be by transfinite induction and appeal to the axiom of choice, as many constructions of nonmeasurable sets do. It will also rely on the following property of the map

$$\pi : K \times K \to [-1,1]$$

given by $\pi(x,y) = x - y$, viz:

Proposition 1.1 *Almost all fibers of π contain perfect sets. In particular, $|\pi^{-1}(t)| = |\mathbb{R}|$ for almost every $t \in [-1,1]$.*

Sketch of the proof. It is simpler to treat the averaging map $f(x,y) = (x+y)/2$, which sends $K \times K$ to $[0,1]$ and has the same behavior as π.

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Suppose $t \in [0,1]$, and for convenience, suppose t is not a rational of the form $p/3^n$. Then t has a unique ternary expansion $t = 0.t_1t_2t_3\ldots_3$; let $N(t)$ be the number of times the digit $t_i = 1$ appears. Then it is readily verified that the fiber $F_t = |f^{-1}(t)| = 2^{N(t)}$ if $N(t)$ is finite, and that $f^{-1}(t)$ is a perfect set if $N(t)$ is infinite. In particular, F_t is perfect for almost every $t \in [0,1]$, but F_t consists of a single point (namely t itself) for $t \in K$ (again assuming t is not a triadic rational). The proof is suggested in Figure 1; note that there are 2 squares above $[1/3, 2/3]$ and only 1 above $[0, 1/3]$ and $[2/3, 1]$. ∎

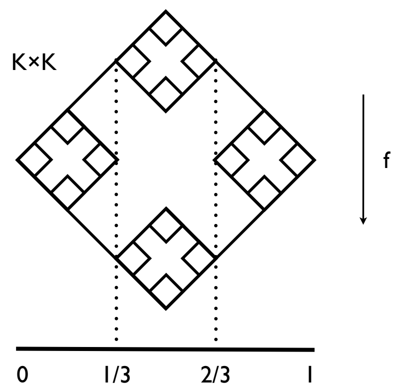

Figure 1. The averaging map from $K \times K$ onto $[0,1]$.

Using this fact, we will show:

Theorem 1.2 *There exists a set $A \subset K$ such that $m^*(A-A) > 0$ but $A-A$ contains no perfect set.*

Proof. Let $\mathfrak{c}$ denote the smallest ordinal with $|\mathfrak{c}| = |\mathbb{R}|$. Then $|\alpha| < |\mathbb{R}|$ for all $\alpha \in \mathfrak{c}$. Since the open and closed subsets of $\mathbb{R}$ themselves have cardinality of the continuum, we can index them by $\mathfrak{c}$, and similarly for the particular types of open and closed sets we consider below.

Thus we let $(P_\alpha : \alpha \in \mathfrak{c})$ denote an enumeration of the perfect subsets of $\mathbb{R}$, and we let $(U_\alpha : \alpha \in \mathfrak{c})$ denote an enumeration of the open sets in $\mathbb{R}$ with $m(U_\alpha) < 2$.

Our goal is to define, by transfinite induction on $\mathfrak{c}$, an increasing sequence of sets $K_\alpha \subset K$ and elements $p_\alpha \in P_\alpha$ such that for all $\alpha \in \mathfrak{c}$:

1. $K_{\alpha+1} - K_{\alpha+1}$ is not contained in U_α; and

2. $K_\alpha - K_\alpha$ does not meet $Z_\alpha = \{p_\beta \,:\, \beta < \alpha\}$.

We then set $A = \bigcup_{\alpha \in \mathfrak{c}} K_\alpha$. By the first property, the outer measure of $A-A$ satisfies $m^*(A-A) \geq 2$; otherwise, $A-A$ would be contained in an open set U with $m(U) < 2$, and we would have $U = U_\alpha$ for some α, contradicting the fact that $K_{\alpha+1} - K_{\alpha+1} \subset A - A$ is not contained in U_α. By the second property, $A - A = \bigcup K_\alpha - K_\alpha$ is disjoint from $\bigcup_{\beta \in \mathfrak{c}} p_\beta$, and hence $A - A$ does not contain any perfect set P_β.

It remains to construct K_α and p_α satisfying the two conditions above. We start the induction with $K_0 = \emptyset$. Whenever we construct K_α, we construct p_α immediately afterwards by choosing any point

$$p_\alpha \in P_\alpha \backslash (K_\alpha - K_\alpha).$$

Such a point exists because $|P_\alpha| = |\mathbb{R}|$, while $|K_\alpha - K_\alpha| \leq |\alpha|^2 < |\mathbb{R}|$.

To construct K_α at a limit ordinal, we let

$$K_\alpha = \bigcup_{\beta < \alpha} K_\beta.$$

It is then immediate by induction that K_α satisfies the two conditions above.

For a successor ordinal $\alpha = \gamma + 1$, we define

$$K_\alpha = K_\gamma \cup \{x, y\}$$

for a suitable pair of points $x, y \in K$.

We will choose (x, y) such that $x - y \notin U_\gamma$; then the first condition will hold. To be more precise, using the proposition above and the fact that $m(U_\alpha) < 2$, we will pick

$$t \in [-1, 1] \backslash U_\alpha$$

such that $F = \pi^{-1}(t)$ contains a perfect set; in particular, $|F| = |\mathbb{R}|$. The first condition then holds, for any $(x, y) \in F$.

We also need to choose (x, y) such that $K_\alpha - K_\alpha$ is disjoint from $Z_\alpha = Z_\gamma \cup \{p_\gamma\}$. Now by induction, $K_\gamma - K_\gamma$ is disjoint from Z_γ. We have also defined p_γ so it is not in $K_\gamma - K_\gamma$, and therefore $K_\gamma - K_\gamma$ is disjoint from Z_α. On the other hand, we have

$$K_\alpha - K_\alpha = (K_\gamma - K_\gamma) + \{0, x, y, x + y, -x, -y, -x - y\}.$$

Thus we need to choose (x, y) such that

$$x \notin E = (K_\gamma - K_\gamma) + Z_\alpha,$$

and similarly for $y, x+y, -x, -y, -x-y$.

Now note that the projection $(x, y) \mapsto x$ is one–to–one on F, and $|E| < |\mathbb{R}| = |F|$. Thus we can easily choose $(x, y) \in F$ so that $\pm x, \pm y \notin E$.

There remains the problem of insuring that $x+y$ and $-x-y$ are not in E, in other words, of insuring that $\pi(x, y) \notin (E \cup -E)$. But this simply requires that we refine our choice of t, so that we also have

$$t \notin E \cup (-E).$$

Since $|E| < |\mathbb{R}|$, most points in $[-1, 1] \backslash U_\alpha$ have this property, and we are done. ∎

Since any set of positive measure contains a perfect set, we have:

Corollary 1.3 *There exists a measurable set $A \subset \mathbb{R}$ such that $A - A$ is not measurable.*

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