Seminar Outline
Dynamics and geometry
Math 290, Spring 1993, Berkeley CA
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1. Teichmüller space and the Teichmüller metric $d(X,Y) = \inf(1/2)\log K(\phi)$, where $\phi : X \to Y$ ranges over all quasiconformal mappings in the appropriate isotopy class.

2. Teichmüller's theorem: If X is compact, or more generally if X is obtained from a compact Riemann surface by removing a finite number of points, then there is an essentially map f minimizing the infimum above. (This map is unique up to conformal automorphisms of X and Y which are isotopic to the identity). The unique map is characterized by the fact that its Beltrami coefficient $\mu = \overline{\partial} f/\partial f$ is of the form $\mu = t\overline{\phi}/|\phi|$, where $|t| < 1$ and ϕ is a holomorphic quadratic differential on X with $\int_X |\phi| < \infty$.

3. Conversely, any integrable holomorphic quadratic differential ϕ determines a *Teichmüller disk* or complex geodesic, by considering the complex structures determined by $t\overline{\phi}/|\phi|$ for all complex t with $|t| < 1$. This set of Riemann surfaces X_t gives a disk in Teichmüller space isometric to the hyperbolic plane.

4. Another way to look at a Teichmüller disk is in terms of the metric determined by a quadratic differential. A holomorphic quadratic differential determines an area form $|\phi(z)||dz|^2$ and hence a metric $|\phi(z)|^{1/2}|dz|$. This is nothing more than the Euclidean metric pulled back from the plane by the map obtained by integrating the 1-form given locally by $\sqrt{\phi(z)}dz$. Since the square-root is determined up to sign, the metric is well-defined, although it has singularities at the zeros of ϕ.

   The underlying conformal structure for this metric is of course the original one. But ϕ also determines a family of metrics by considering an affine change of coordinates. Exercise: write down a formula for this family of metrics.

5. Quadratic differentials, polygons and billiards. A quadratic differential determines a flat metric (with standard singularities) and a foliation

by equidistant lines. Conversely, such a structure determines a Riemann surface and a quadratic differential. A nice example is to take a rectangangle foliated by horizontal lines and double it; this gives a quadratic differential on the sphere with 4 simple poles, and represents a cotangent vector to the Teichmüller space of the 4 times punctured sphere.

Thus understanding billiards in a rectangle can be related to the geodesic flow for a quadratic differential on the sphere, or on its 2-fold branched cover the torus. Similar, billiards on an L-shaped table leads to a quadratic differential on a sphere with six singularities, or to one on its double cover, a surface of genus two. One can imagine billiards following the leaves of the foliation determined by the differential.

The operations of affinely squeezing the table (which does not change the dynamics) and varying the slope of the trajectory (which does) trace out a Teichmüller disk of conformal structures.

6. The space $Q(X)$, the set of ϕ with $||\phi|| = \int_X |\phi| < \infty$, is the cotangent space to Teichmüller space at X; its norm is the infinitesimal form of the Teichmüller cometric.

7. Examples of Teichmüller mappings: (a) affine maps between tori. (b) Let $f : X_1 \to X_2$ be a Teichmüller map between tori, and let $Y_1 \to X_1$ be a branched covering of X_1 by a surface of genus $g > 1$, branched over $E_1 \subset X_1$. Then f lifts to a Teichmüller map between Y_1 and Y_2, where Y_2 is the corresponding covering of X_2 branched over $E_2 = f(E_1)$. The associated quadratic differential on Y_1 has zeros at the critical point of π.

   A special case of this example is obtained by letting E_i be the points of order n in the group law on X_i, and letting Y_i be the covering determined by $H_1(X_i - E_i, \mathbb{Z}/m)$. In this case everything is equivariant and letting X_2 vary we get an isometry from $\mathcal{M}_1$ into $\mathcal{M}_g$, where g is the genus of Y_i. These *complex geodesics of finite volume on moduli space* seem quite rare but are not yet classified.

8. The mapping class group of a finite volume hyperbolic surface. Royden's theorems: Kobayashi = Teichmüller metric; the mapping class group, isometry group and complex automorphism group of Teichmüller space all coincide (in dimension two or more).

9. The Kobayashi and Teichmüller metric coincide. The proof appeals to the Ahlfors principle, which says that for a holomorphic map of the unit disk into a complex hermitian manifold, the curvature of the induced metric is at least as negative as the section curvature tangent to the image. Also, every ultrahyperbolic metric (of curvature less than -1) lies below the hyperbolic metric. For Teichmüller space, every complex tangent plane is tangent to a Teichmüller disk, which has constant negative curvature. Then one finds the Teichmüller disks give the extreme value for the Kobayashi metric definition.

   Ahlfors' principle is related to the fact that every analytic submanifold is a *minimal surface*. For maps of the disk to $\mathbb{C}^n$ this implies the induced curvature (for the Euclidean metric) is non-positive.

   Exercise. Check this directly (it reduces to the Cauchy-Schwarz inequality).

   The proof of minimality is by the fact that the integral of the Kähler form over the disk gives the area, whereas in general it gives a lower bound for the area; using the fact that the Kähler form is closed, we see that the area of the disk is only increased if it is changed by a homology.

10. Theorem (Ahlfors' result on ultrahyperbolic metrics). Let $\rho_1(z)|dz|$ be a metric on the disk of variable curvature less than -1, and let $\rho(z)|dz|$ be the hyperbolic metric. Then $\rho_1(z) \leq \rho(z)$ everywhere.

    Remark: As you increase any metric its curvature tends to zero. So as you descrease a metric (make points closer together) its curvature becomes more pronounced (positively or negative). The theorem is a sort of converse.

    Proof. We will use the formula for the curvature

$$K(\rho) \quad = \quad -\frac{\Delta \log \rho}{\rho^2}.$$

    By a well-known trick (shrinking the disk slightly) we may assume $\rho_1/\rho \to 0$ at the boundary of the disk (e.g. arrange that the disk has finite diameter in the ρ_1 metric). Then this ratio assumes its maximum at some point p. It suffices to check that $\rho_1/\rho \leq 1$ at this point. But the Laplacian at a critical point is just the trace of the Hessian, which must have negative eigenvalues. Thus $\Delta \log(\rho_1/\rho) \leq 0$ at p. Combining this with the formula for curvature and the assumption that $K(\rho_1) \leq -1$ yields the theorem. ∎

Exercise: Prove for any conformal metric on the sphere with curvature greater than one, distances are shorter than in the metric of constant curvature one.

11. The mapping class group coincides with the isometry group. The proof is to show that the "most wrinkled" (least smooth) points on the unit ball in $Q(X)$, when projectivized, give a model for X (the dual bi-canonical embedding).

    More precisely, for each point p on X of genus $g \geq 3$, there is a unique quadratic differential ϕ_p (up to a complex multiple) with a zero of order at least $3g-4$ at p. The other zeros have lesser order, so ϕ_p determines p. The Riemann surface X is isomorphic to the projectivization of the set of ϕ_p in $Q(X)$. On the other hand, these ϕ_p can be distinguished by the norm, so the norm on $Q(X)$ determines X.

    This argument should be compared to the classical theorem of Banach and Stone, which asserts that for compact Hausdorff spaces X and Y, an isometry from $C(X)$ to $C(Y)$ naturally determines a homeomorphism from X to Y. (The space X can be recovered as the projectivization of the set of extreme points of the unit ball in $C(X)^*$ with respect to the weak* topology.)

    Another proof that every isometry comes from a modular transformation has been given by Ivanov, by considering automorphisms of a simplical complex attached to the collection of all systems of disjoint simple closed curves on a surface.

12. Moduli space $\mathcal{M}_g$ is the quotient of Teichmüller space by the mapping class group; it has the structure of a complex orbifold, and it can be shown to be a quasi-projective variety.

    Mumford's Theorem: Every compact subset of moduli space is contained in one of the form $\{[X] : \ell(X) \geq \epsilon > 0\}$, where $\ell(X)$ is the length of the shortest geodesic on X in the hyperbolic metric.

    In the case of a surface of genus 1, this reduces to Mahler's compactness criterion for a family of lattices in $\mathbb{R}^2$. (Mahler's criterion states that the space of lattices in $\mathbb{R}^n$ with a lower bound on the length of the shortest vector and an upper bound on the covolume is compact.)

An "elementary" proof of Mumford's Theorem is the following. Given a sequence X_i with $\ell(X_i) > \epsilon$, cut X_i up into pairs of pants. We can assume the lengths of the cuffs are bounded above and below, so after passing to a subsequence the constituent pairs of pants converge. There are only a finite number of gluing patterns, so this converges as well. Finally the twist parameters can be chosen to converge too, QED.

Actually a similar proof can be given by covering the surface by a finite number of overlapping balls of definite size, which can only intersect in a finite number of combinatorial patterns. The latter proof generalizes to higher dimensional situations.

It is known that a pair of pants decomposition of a surface of genus g exists with cuffs of length $O(g)$. It is also known that one cannot do better than $C\sqrt{g}$; see Buser's book, section on the hairy torus. The idea is to take two square tori and join the nth roots of unity in the group of very thin tubes. One must cut along these tubes in any reasonable pair of pants decomposition. Somewhere in the remaing decomposition is a curve which is nontrivial on the torus with the nth roots of unity filled in. This one has length at least $c\sqrt{g}$.

13. Some differences between Teichmüller space and hyperbolic space. (a) There exist automorphisms with positive translation distance which is not achieved (a pseudo-Anosov supported on a subsurface). (b) There exist a pair of distinct geodesics through a single point which do not diverge (Masur's example of a pair of Strebel differentials with different heights.) This is a positive-curvature-like phenomenon. (c) The end of moduli space carries the full fundamental group for Teichmüller spaces of dimension two or more. This belies the thick-thin decomposition.

14. Topologically Teichmüller space is a cell; that is, $Teich(S) \cong \mathbb{R}^{6g-6}$. One way to see this is to use quadratic differentials as angular coordinates and stretch factors as radial coordinates.

15. The mapping class group of a closed surface is nothing more than the outer automorphism group $\mathrm{Out}(\pi_1(S)) = \mathrm{Aut}(\pi_1(S))/\,\mathrm{Inner}(\pi_1(S))$. For surfaces with punctures, this is not quite true: one must restrict to

automorphisms which preserve the peripheral structure of the group, i.e. which preserve loops around the punctures.

Exercise. Give an automorphism of the fundamental group of the triply-punctured sphere which does not have this property.

16. The mapping class group acts properly discontinuously on Teichmüller space.

    To see this, it is useful to know that there exist $9g-9$ simple loops on S whose hyperbolic lengths on $X \in \text{Teich}(S)$ determine X uniquely. This can be proved by taking $3g-3$ loops in a pair of pants decomposition, then for each of these two more, each meeting just that cuff of the pair of pants decomposition and differing by a Dehn twist around that cuff. From this additonal pair of lengths one can reconstruct the twist parameters for Fenchel-Nielsen coordinates. (See Imayoshi and Taniguchi).

    Now the proper discontinuity follows from the more general observation of Wolpert, that the set of Y in $\text{Teich}(S)$ which have the same length spectrum as X is discrete. This is proved by following the $9g-9$ lengths above as one moves from X to Y. Each length moves only by a bounded factor, given in terms of $d(X,Y)$. This pins the new values for the lengths down to a finite set.

17. Classification of mapping classes. Every mapping class Φ is either (a) reducible, (b) of finite order, or (c) pseudo-Anosov. This classification corresponds to:

    (a') translation distant not achieved; (b') achieved but equal to zero; and (c') achieved and positive.

    Clearly these possibilities are mutually exclusive.

    To see (a') implies (a): there exist Riemann surfaces X_n, such that $[X_n]$ tends to infinity in moduli space but the translation distance is bounded; then consider the short geodesics on X_n to find a reduction of the mapping class.

    We must actually show that $[X_n]$ bounded in moduli space implies the translation distance is achieved. This uses proper discontinuity of the mapping class group.

That (b') implies (b) is obvious, since the group of conformal automorphisms is finite.

In case (c'), we use the fact that Teichmüller space is a *straight space*: there exists a unique infinite geodesic joining any two points. If the translation distance is achieved at X, we claim the geodesic γ connecting X to $\Phi(X)$ is stabilized by Φ. Indeed, consider the midpoint Y of the segment J from X to $\Phi(X)$; if $\Phi(J)$ were not simply the prolongation of γ, then the distance from Y to $\Phi(Y)$ would be less than that from X to $\Phi(X)$.

18. For a pseudo-Anosov class, the extremal map from X to $\Phi^2(X)$ is the composition of the one from X to $\Phi(X)$ with the one from $\Phi(X)$ to $\Phi^2(X)$. From this it follows easily that the initial and terminal quadratic differentials of Φ agree. Thus we can assert:

    Theorem. A pseudo-Anosov mapping class has a representative which preserves a pair of orthogonal singular measured foliations and stretches them by a factor of $\sqrt{K}$ $(1/\sqrt{K})$ respectively.

    Corollary. A mapping class is pseudo-Anosov if and only if for every nontrivial closed curve γ and every Riemannian metric, the length of $\Phi^n(\gamma)$ grows like K^n for some $K > 1$.

    This constant K is independent of γ and is determined by the length of the closed Teichmüller geodesic on moduli space determined by Φ.

19. A further and more constructive discussion of pseudo-Anosov mappings would lead into the theory of train tracks, geodesic currents and Thurston's compactification of Teichmüller space.

20. A brief aside on 3-manifold that fiber over the circle with fiber a closed torus. The manifold admits a Euclidean, Nil, or Solve structure as the monodromy is of finite order, parabolic or hyperbolic.

    For fibers of genus $g > 1$, the finite order case implies an $\mathbb{H} \times \mathbb{R}$ structure, while the reducible case implies the manifold is torus-reducible.

    Our main concern with be to show that the pseudo-Anosov case corresponds to a hyperbolic structure.

21. Kleinian groups and hyperbolic 3-manifolds. The limit set, domain of discontinuity. Finitely generated groups and Ahlfors' finiteness theo-

rem. The general deformation theory; Sullivan's no invariant line fields theorem; the measure zero conjecture.

Examples: Schottky groups, Fuchsian groups, the right angled dodecahedron.

22. Quasifuchsian groups and hyperbolic structures on $S \times \mathbb{R}$. Using quasiconformal deformations, one obtains a natural map

$$\eta : Teich(S) \times Teich(\overline{S}) \to \mathrm{Hom}(\pi_1(S), \mathrm{Aut}(\mathbb{H}^3))/\mathrm{conjugation} \quad = \quad V(\pi_1(S)).$$

(There are two reasons to mod out by conjugation: first, one must pick a basepoint to define π_1, and change of this basepoint amounts to inner automorphism; and second, the group $\eta(X, Y)$ is itself determined only up to conjugation in the Lie group $\mathrm{Aut}(\mathbb{H}^3)$). This map has the following properities:

(a) $\eta(X, Y) = \Gamma(X, Y)$ is a Kleinian group, that is a discrete subgroup of $\mathrm{Aut}(\mathbb{H}^3)$.

(b) The limit set of $\Gamma(X, Y)$ is a Jordan curve, in fact a quasicircle, with two componentary disks Ω_X and Ω_Y.

(c) The quotients $\Omega_X/\Gamma(X, Y)$ and $\Omega_Y/\Gamma(X, Y)$ are isomorphic to X and Y. Moreover the isomorphism comes with a natural marking of $\pi_1(X)$ and $\pi_1(Y)$ that provides an inverse for the map η.

(d) The map η is injective, holomorphic and open.

(e) The image of Teichmüller space embedded in its product by $X \mapsto \Gamma(X, \overline{X})$ maps under η isomorphically to the set of Fuchsian groups uniformizing S.

23. The mapping class group acts on the representation variety and on the product of Teichmüller spaces in a compatible way. That is, any mapping class on S gives one on $\overline{S}$ tautologically, and acting by these two simultaneously gives a map $\Phi : (X, Y) \mapsto (\Phi(X), \Phi(Y))$. This map is holomorphic, and under η it becomes the restriction of an automorphism of $V(\pi_1(S))$ induced by $\mathrm{Out}(\pi_1(S))$.

This action also preserves the subspace $AH(S)$ of discrete faithful representations. This subspace parameterizes all marked hyperbolic 3-manifolds with the homotopy type of S.

24. The complete topological picture of $AH(S)$ and its relations to $T(S) \times T(\overline{S})$ is not known, but conjecturally the latter is dense in the former. For the case of a punctured torus, the product of Teichmüller spaces can be thought of as $\mathbb{H} \times \mathbb{L}$, where $\mathbb{L}$ is the lower half-plane. The conjectural closure of this product in $AH(S)$ is obtained from $\overline{\mathbb{H}} \times \overline{\mathbb{L}}$ by deleting S^1 embedded diagonally in $S^1 \times S^1$. Here $S^1 = \mathbb{R} \cup \{\infty\}$ compactifies $\mathbb{H}$ and $\mathbb{L}$. This gives a good picture of the action of a hyperbolic mapping class: its fixed points are of the form (λ, μ) where $\lambda \neq \mu$ are its two fixed points on S^1.

25. Theorem. Let $\phi : S \to S$ be a mapping class of infinite order. Then M_ϕ, the mapping cylinder of ϕ, admits a hyperbolic structure if and only if ϕ has a fixed point in $AH(S)$.

    Exercise. Show any reducible mapping class fails to have a fixed point in $AH(S)$. What happens if the mapping class is of finite order?

26. Bers' slice and compactifications of Teichmüller space. Cusps and totally degenerate groups.

27. The double limit theorem in the pseudo-Anosov case:

    Theorem. Given any X and Y in $\text{Teich}(S) \times \text{Teich}(\overline{S})$, and a pseudo-Anosov mapping class ψ, the groups $\Gamma_n = \Gamma(\psi^n(X), \psi^{-n}(Y))$ form a precompact subset of $AH(S)$.

    The proof uses pleated surfaces. Actually, we will see that this sequence has a unique limit point.

28. Each Γ_n can be equipped with a K-quasiconformal map ϕ_n inducing the mapping class ψ_n; here K is equal to $\max(d(X, \psi(X)), d(Y, \psi(Y))$ and is independent of n. Passing to a subsequence, we can assume $\Gamma_n \to \Gamma_\infty$ and $\phi_n \to \phi_\infty$. But the limit set of Γ_infty is the whole sphere, so by Sullivan's theorem ϕ_∞ is conformal. Then the group generated by Γ_∞ and ϕ_∞ is equal to $\pi_1(M_\psi)$. By topological results, the quotient hyperbolic manifold is homeomorphic to M_ψ, so we have shown that M_ψ is hyperbolic.

29. The renormalization viewpoint. How to find the fixed point of a hyperbolic transformation? Using this idea, we recast the proof in two steps: first, form the limit $\Gamma' = \lim \Gamma(X, \psi^{-n}(Y))$ in a Bers slice (logically

we must take a limit along a subsequence.) Then, we consider the sequence of groups $\Gamma'_n = \psi^n(\Gamma')$. These groups are all isomorphic; we are just viewing the same dynamical system from a changing perspective.

Then, the hard moment in the proof is to show that Γ'_n converges. Assuming this, we may now consider *geometric limits* of Γ'_n, show they all have bounded injectivity radius, deduce that they are rigid on the sphere at infinity, and therefore the quasiconformal automorphisms ϕ'_n converge to conformal maps.

30. **Ahflors' finiteness theorem and Bers' area theorem.** The Teichmüller space of the quotient Riemann surface for a finitely generated Kleinian group is finite dimensional (Ahlfors); its area is also finite (Bers). The second statement is stronger because the first does not rule out the possibility of infinitely many triply-punctured spheres.

31. **Ahlfors' measure zero conjecture.** Let Λ be the limit set of a finitely generated Kleinian group. Then either $\Lambda = \widehat{\mathbb{C}}$ and the action of Γ on Λ is ergodic, or the area of Λ is equal to zero.

    This conjecture has been proved for any Kleinian group isomorphic to the fundamental group of a surface, by Bonahon and Thurston. The method of proof is to construct pleated surfaces marching off into the geometrically infinite ends. This property is equivalent to the existence of "simple" geodesics (i.e. those that become simple closed curves under the homotopy equivalence to a surface) arbitrarily deep in those ends.

32. **Pleated surfaces and projective structures.** A surface plus a bending measure determines a group of hyperbolic motions and a developing map of the surface. This is related to the fact that a projective structure on a surface determines both a group and a developing map.

    The projective structure on $\Omega_{\overline{X}}$ provides a realization of Teichmüller space as a bounded domain in $Q(\overline{X})$.

33. Bers' slices: boundedness, scarcity of cusps, existence of totally degenerate groups.

Bibliographical notes. Royden's results appear in [Roy]. Bers' approach to Thurston's classification of mapping classes is in [Bers]; see also [FLP]. An excellent recent book on Teichmüller theory is [IT]; see also [Bus]. Thurston's construction of hyperbolic structures on 3-manifolds which fiber over the circle is presented in [Th]; see also Sullivan's Seminaire Bourbaki talk.

References

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