Course Outline
Complex Manifolds
Math 241, Spring 1996, Berkeley CA
C. McMullen

Texts:
Forster, *Lectures on Riemann Surfaces*, Springer-Verlag
Griffiths and Harris, *Principles of Algebraic Geometry*, Wiley

Also recommended:
Cornalba et al, *Lectures on Riemann Surfaces*, World Scientific
Griffiths, *Lectures on Algebraic Curves*, AMS

1. Holomorphic functions in one and several variables. Definition of a complex manifold. Canonical splitting of the complexified tangent and cotangent space.
$$T_{\mathbb{R}} \subset T \otimes \mathbb{C} = T' \oplus T'',$$
where $T_{\mathbb{R}}$ is the real tangent space at a point, $T'' = \langle \partial/\partial \overline{z}_i \rangle$ annihilates holomorphic functions and $T' = \langle \partial/\partial z_i \rangle$ annihilates antiholomorphic functions. On $\mathbb{C}$,
$$Df(w) = \frac{\partial f}{\partial z} w + \frac{\partial f}{\partial \overline{z}} \overline{w}.$$
Quasiconformal maps. Stone-Weierstrass theorem: a continuous function can be approximated by a polynomial in z and $\overline{z}$.

2. Examples of complex manifolds: domains in $\mathbb{C}^n$, branched coverings, $\mathbb{C}^n/\Lambda$, $(\mathbb{C}^n - 0)/(z \mapsto \lambda z)$, $\mathbb{P}^n$, maximal analytic continuation of a germ in $\mathbb{C}$, hypersurfaces such as $x^n + y^n = 1$ in $\mathbb{C}^2$, surfaces in space (more generally Riemannian surfaces), Fuchsian and Kleinian groups. Minimal surfaces: a surface $\Sigma \subset \mathbb{R}^2$ is minimal iff it is nonpositively curved and the Gauss map is holomorphic.

3. Cauchy's integral formula: $\int_\gamma f(z)dz$. Intrinsic point: $\omega = f(z)dz$ is a *closed* 1-form, since $d\omega = (df/d\overline{z})d\overline{z} \wedge dz = 0$. Removable singularities theorem.

4. Completing Riemann surfaces: if every end of X has a neighborhood isomorphic to Δ^*, then there is a *canonical* compactification Y of X

such that $Y - X$ is finite. Use removable singularities to prove completion is *canonical*.

Example: two-sheeted branched covers over $\widehat{\mathbb{C}}$.

5. When can a Riemann surface be embedded in the sphere? A: whenever it has no handles. Example: the Schottky construction of planar coverings; Schottky uniformization. Example: a classical Schottky group (using reflections in disjoint circles).

6. Solution to the $\overline{\partial}$ equation in $\mathbb{C}$; $\phi = (1/2\pi i)(-1/z)$ satisfies $\overline{\partial}\phi = \delta(z) d\overline{z}$ in the sense of distributions, so

$$f(z) = g * \phi = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{g(w)\, dw \wedge d\overline{w}}{w - z}$$

satisfies $\overline{\partial} f = g(z) d\overline{z}$, for any smooth compactly supported $g(z)$.

7. Mappings between Riemann surfaces; basic structure theorem (f is constant or $w = f(z) = z^n$ for appropriate charts). Two proofs: (a) apply Riemann mapping theorem to preimage of a small ball under f; (b) use $f(z) = z^n h(z)$, $h(0) = 1$, to define $f(z)^{1/n}$ and show it is invertible.

8. Corollaries: $f : X \to Y$ (if nonconstant) is open, with discrete fibers. If X is compact, then f is surjective. A bounded holomorphic function on $\mathbb{C}$ extends to $\widehat{\mathbb{C}}$ and is therefore constant. Thus $\mathbb{C}$ is algebraically closed (consider $1/p(z)$).

9. Hartog's phenomenon, Weierstrass preparation theorem, functions on $\mathbb{C}^n$ as parameterized functions on $\mathbb{C}$. Riemann extension theorem in one and several variables.

10. Meromorphic functions on complex manifolds (locally given by $(U_\alpha, f_\alpha, g_\alpha)$. The example of z_2/z_1 on $\mathbb{C}^2$; blowing up. $K(\mathbb{P}^n) = \mathbb{C}(Z_1/Z_0, \ldots, Z_n/Z_0)$ (proof for $n = 1$).

11. The residue theorem: if X is compact, ω is a meromorphic 1-form, then $\sum_P \operatorname{Res}_P(\omega) = 0$.

12. General theory of covering spaces, branched coverings, proper maps. Coverings of the punctured disk.

    Theorem: given X compact, $E \subset X$ finite, and $G \subset \pi_1(X - E)$ of finite index, there is Riemann surface Y and a proper holomorphic map $\pi : Y \to X$, unique up to isomorphism over X, such that $Y - \pi^{-1}(E)$ is isomorphic to the covering space of $X - E$ corresponding to G.

13. Universal coverings of Riemann surfaces are isomorphic to $\mathbb{H}$, $\mathbb{C}$ or $\widehat{\mathbb{C}}$.

14. The category of compact Riemann surfaces and the category of fields finite over $\mathbb{C}(z)$. The trace or pushforward $\mathrm{tr} : K(Y) \to K(X)$ for a finite extension $K(Y)/K(X)$, and for a proper map $f : Y \to X$.

15. The resultant $R(f, g)$ of two polynomial $f(z)$ and $g(z)$ of degrees d and e is a polynomial in the coefficients that vanishes iff there are monic polynomials $r(z)$, $s(z)$ of degree $e-1$ and $d-1$ such that $rf + sz = 0$. It is given by a determinant by looking at the span of $z^i f$ and $z^j g$. The discriminant $D(f) = R(f, f')$ vanishes iff f has a multiple root.

16. Constructing X from $K(X)$: express $K(X)$ as $k(f)$, where $k = K(\widehat{\mathbb{C}}) = \mathbb{C}(z)$. Let $P(f, z)$ be the irreducible polynomial satisfied by f; then X can be obtained by completing $\{(y, z) : P(y, z) = 0\} \subset \mathbb{C} \times \widehat{\mathbb{C}}$. By considering pushforwards one finds $\deg K(X)/K(Y)$ is always at most $\deg(X/Y)$, so $K(X) = k(f)$.

17. The Fermat curve $X(n)$ described by $x^n + y^n = 1$. Projection by x to $\widehat{\mathbb{C}}$ is branched over $x^n = 1$, and since $y = \sqrt[n]{1 - x^n}$ each branch point has multiplicity $n - 1$. For large x, we have $x^n \approx y^n$, so there are n sheets over infinity. Thus $\chi(X(n)) = 2n - n(n-1) = 3n - n^2$.

18. Theorem: if $f : X \to Y$ is nonconstant, then $g(X) \geq g(Y)$. Corollary: The Fermat equation $f^n + g^n = 1$ has no solutions in $\mathbb{C}(z)$ except with f and g constant, if $n > 2$. It also has no entire solutions for $n > 2$, and no meromorphic solutions for $n > 3$, since the universal cover of $X(n)$ is then the disk.

    For $n = 3$ there are meromorphic solutions coming from the Weierstrass $\wp$-function. For $n = 2$ you can take $f(t) = t^2 - 1$, $g(t) = 2t$, $h(t) = t^2 + 1$. Geometrically this corresponds to projecting the circle to the vertical axis from the point $(-1, 0)$.

19. Sheaves. Basic examples: $\mathcal{O}$, $\mathcal{O}^*$, $\mathcal{M}$, $\mathcal{M}^*$ (holomorphic and meromorphic functions), Ω^p (holomorphic forms), C^∞, $\mathcal{A}^p$ (smooth p-forms), $\mathcal{Z}^p$ (closed p-forms), $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{C}$ (locally constant functions).

    Presheaves that are not sheaves: $\mathcal{B}$ ("boundaries" — exact forms), $H^p(U)$ (cohomology groups).

20. A section of a sheaf is determined by its germs. L'espace etalé $|\mathcal{F}|$ of a presheaf $\mathcal{F}$.

21. Analytic continuation. Any germ of an analytic function $f \in \mathcal{F}_a$ on X has a unique maximal analytic continuation $g : Y \to \mathbb{C}$, with $\pi : (Y, b) \to (X, a)$ a holomorphic local homeomorphism sending g_b to f_a.

22. Elliptic functions: if $X = \mathbb{C}/\Lambda$, then $K(X) = \mathbb{C}(\wp, \wp') \cong \mathbb{C}(x)[y]/(y^2 = 4x^3 + ax + b)$. Proof: $\wp : X \to \widehat{\mathbb{C}}$ presents the sphere as the quotient of X under the involution $z \mapsto -z$. Thus any even function on the torus descends to a function on the sphere, where is becomes a rational function of $\wp$. Since $\wp'$ is odd, and any function is a sum of and even one and an odd one, we find $\wp$ and $\wp'$ generate $K(X)$.

23. The uniformization of elliptic curves. Theorem: given any degree two covering $X \to \widehat{\mathbb{C}}$ branched over 4 points, there is a lattice Λ such that $X = \mathbb{C}/\Lambda$. In fact Λ is generated by the periods of

    $$\omega_0 = \frac{dz}{\sqrt{(z-a)(z-b)(z-c)}}$$

    if the branch points are $\{a, b, c, \infty\}$.

    Proof: It suffices to find a nowhere zero holomorphic 1-form ω on X. For then we get a map $\delta : \widetilde{X} \to \mathbb{C}$ by integrating its lift to the universal cover; this map is an isometry from $|\omega|$ to $|dz|$, so it is a covering map; therefore it is an isomorphism. The action of $\pi_1(X)$ on $\widetilde{X}$ is clearly conjugate under δ to the action of translation by the periods, so the latter are linearly independent over $\mathbb{C}$ and we are done.

    But the multivalued form ω_0 becomes a single-valued form ω on the 2-sheeted cover of $\widehat{\mathbb{C}}$ branched over a, b, c, ∞, so we are done.

    Caveat:

    $$\wp^{-1}(p) = \int^p \frac{dz}{\sqrt{(z-a)(z-b)(z-c)}}.$$

This is the origin of the word 'elliptic integral'

24. Trigonometric functions can be thought of as arising by a similar procedure. For example, we wish to show the universal cover of the $(2, 2, \infty)$-orbifold is the plane. That is, we want to show $\widetilde{X} \cong \mathbb{C}$ where $f : \widetilde{X} \to \mathbb{C}$ is a regular, infinite-sheeted branched covering, branched with order 2 over -1 and 1. To this end consider the form

$$\omega_0 = \frac{dz}{\sqrt{(1-z)(1+z)}}.$$

When pulled back to $\widetilde{X}$ and integrated it establishes the desired isomorphism, and we have

$$\sin^{-1}(p) = \int^p \frac{dz}{\sqrt{(1-z)(1+z)}}.$$

However one should be careful: for coverings of noncompact surfaces, the integral of ω does not automatically give an isomorphism to $\mathbb{C}$; e.g. consider $\omega = e^z dz$ on $\mathbb{C}$.

25. Puiseux series: the germ of an algebraic function near $z = 0$ can always be expressed as a Laurent series in $\zeta = z^{1/d}$, where d is the degree of the function.

As a corollary one can see that the links of singularities of hypersurfaces in $\mathbb{C}^2$ are iterated torus knots. The point is that the knot can be parameterized by $x = \zeta^{d_0}$, $y = \sum_1^\infty a_i \zeta^{d_i}$, where $gcd(d_0, d_1, \ldots) = 1$. The successive jumps in $gcd(d_0, \ldots, d_k)$ give the iterations.

26. *Avec les séries de Puiseux,*
Je marche comme sur des oeufs.
Il s'ensuit que je les fuis
Et me refugie dans la nuit...

—A.D.

Variante pour le dernier vers:

Il s'ensuit que je les fuis
Comme un poltron que je suis.

27. The cotangent space in terms of stalks: it can be written $T_p^*X = m_p/m_p^2$, where m_p are the germs of smooth functions vanishing at p. Then $(df)_p = (f - f(p))$.

28. A holomorphic 1-form is closed, and a closed (1,0)-form is holomorphic.

    The Hodge $*$ operator. On an oriented n-dimensional real vector space V with an inner product, the operator

$$* : \bigwedge^p V \to \bigwedge^{n-p} V$$

    is defined so that

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle \text{vol}.$$

    Here an orthonormal basis for the p-forms is obtained by wedging together elements of an orthonormal basis for V.

    On the complexification of V one has a Hermitian inner product, and the complex linear extension of $*$ satisfies

$$\alpha \wedge *\bar{\beta} = \langle \alpha, \beta \rangle \text{vol}.$$

    From this we obtain a hermitian inner product on p-forms on an oriented Riemannian manifold by

$$\langle \omega_1, \omega_2 \rangle \quad = \quad \int_X \omega_1 \wedge *\bar{\omega}_2.$$

29. The composition of Hodge $*$ with complex conjugation gives $\bar{*}$. On a Riemann surface, the $\bar{*}$ operator interchanges $\Omega(X)$ and $\overline{\Omega}(X)$, the spaces of holomorphic and antiholomorphic forms, and $*^2 = \bar{*}^2 = -1$ on 1-forms on a surface.

30. Harmonic forms on a compact manifold. A p-form is *harmonic* if $(dd^* + d^*d)\omega = 0$, where d^* is the adjoint to exterior d with respect to the inner products introduced above. Equivalently, ω is harmonic if it is closed ($d\omega = 0$) and co-closed ($d * \omega = 0$). This is because $d * \omega = \pm * d * \omega$ (as can be seen by integration by parts), and

$$\langle \omega, (dd^* + d^*d)\omega \rangle = \langle d\omega, d\omega \rangle + \langle d * \omega, d * \omega \rangle.$$

    A closed form is harmonic iff it formally minimizes its L^2-norm in its cohomology class; this is because $d * \omega = 0$ if and only if $\langle \omega, df \rangle = 0$ for all $(p-1)$-forms f.

31. The *Hodge Theorem* asserts that for a compact oriented Riemannian manifold, we have an isomorphism

$$\mathcal{H}^p_\Delta(X) \cong H^p_{\rm DR}(X),$$

between the space of harmonic forms for the Hodge Laplacian $\Delta = dd^* + d^*d$ and the de Rham cohomology of X. The arguments already given show this map is injective, since a harmonic form satisfies $\|\omega + df\| \geq \|\omega\|$.

32. The harmonic 1-forms on a Riemann surface are exactly $\Omega(X) \oplus \overline{\Omega}(X)$, and any such is locally $\omega = df$ with f a harmonic function.

33. Periods: a 1-form is exact if and only if all its periods vanish. A harmonic function is determined by its periods. Thus we have

$$\Omega(X) \oplus \overline{\Omega}(X) \hookrightarrow H^1(X) = H_1(X)^*,$$

so if X has genus g then $\dim \Omega(X) \leq g$. Example: $X = \mathbb{C}/\Lambda$; equality holds since $\Omega(X)$ contains dz.

So as soon as we can exhibit g holomorphic forms on X, we have: $\dim \Omega(X) = g$; and every cohomology class is represented by a harmonic form.

34. Linear differential equations. $F : X \to \mathbb{C}^n$ satisfying $dF = M \cdot F$, where M is an $n \times n$ matrix of holomorphic 1-forms. This can also be thought of as a holomorphic *connection* (gl_n-valued 1-form) on the trivial bundle $X \times \mathbb{C}^n$; since $dM = 0$ it is flat. If M is contained in a subalgebra of gl_n then the structure group is suitably reduced.

Example: $f'' + \phi f = 0$ corresponds to $M = \left(\begin{smallmatrix} 0 & 1 \\ -\phi & 0 \end{smallmatrix}\right)$ and $F = (f, f')$. Since $\operatorname{tr} M = 0$ we see the *Wronskian* $fg' - f'g$ of any pair of solutions is constant, correspond to the fact that the determinant is preserved ($M \in {\rm sl}_2$).

35. Local solvability. From $dF = M \cdot F$ on the disk $D(0,1) \subset \mathbb{C}$ we get the recursion relations for power series:

$$F_{i+1} = \frac{1}{i+1} \sum_{j+k=i} M_j F_k.$$

Given the initial condition F_0 the above recursively determines F_i.

Now assume $\|M_j\| \leq M$ for all j (which is true if M is holomorphic on a slightly larger disk). Then we claim for any $\lambda = (1+\epsilon) > 1$, there is a C such that $\|F_i\| \leq C\lambda^i$. To see this, choose δ so $\delta(\lambda^0 + \ldots + \lambda^i) < \lambda^{i+1}$, choose I so $M/(i+1) < \delta$ for $i > I$, and choose C so $a_i \leq C\lambda^i$ for $i \leq I$. Then by induction we have

$$|F_{i+1}| \leq \frac{M}{i+1}(|F_0| + \ldots + |F_i|) \leq \delta C(\lambda^0 + \ldots \lambda^i) \leq C\lambda^{i+1}.$$

36. Monodromy of differential equations. In general the equation $dF = M \cdot F$ has global solutions only on the universal cover $\widetilde{X}$, and we get a *monodromy representation* $\rho : \pi_1(X) \to GL(V)$, where V is the (finite-dimensional) space of solutions.

    Example: on $X = \Delta^*$ consider the equation $f(z) - zf'(z) + z^2 f''(z) = 0$. It has a basis of solutions $f_1(z) = z$ and $f_2(z) = z \log z$. Analytically continuing once around the puncture we find these are transformed to $g_1 = f_1$ and $g_2 = f_2 + 2\pi i f_1$, so the monodromy matrix is $\left(\begin{smallmatrix} 1 & 0 \\ 2\pi i & 0 \end{smallmatrix}\right)$

37. The *Schwarzian derivative* of a conformal map f between regions on the sphere is a holomorphic quadratic differential given by $Sf = Sf(z)dz^2$ where
$$Sf(z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2.$$
    It is connected to differential equations by the fact that $Sf = \phi$ if and only if $f = g_1/g_2$, where g_1, g_2 are solutions to the equation
$$g''(z) + \frac{1}{2}\phi(z)g(z) = 0.$$
    For example if $\phi = 0$ then $g_i(z) = a_i z + b_i$ and f is a Möbius transformation. In general Sf measures the deviation of f from being Möbius.

38. Cocycles. The Schwarzian can be compared to the *log derivative* $Df(z) = \log f'(z)$, and the *nonlinearity* $Nf(z) = (f''(z)/f'(z))dz$. All three are cocycles; that is,
$$D(f \circ g) = Dg + g^*(Df)$$
    and similarly for Nf and Sf; here it is crucial to think of these objects as forms.

39. Uniformization of the triply-punctured sphere. Let $\pi : \mathbb{H} \to X = \widehat{\mathbb{C}} - \{0, 1, \infty\}$ be the uniformizing map. Then π^{-1} is well-defined up to composition with a Möbius transformation, so $\phi = S\pi^{-1}$ is a holomorphic quadratic differential on X. Near the punctures, π^{-1} behaves like $\log(z)$, and $S\log(z) = (1/2)dz^2/z^2$, so we can deduce (using S_3 symmetry of X) that

$$\phi = \frac{z^2 - z + 1}{2z^2(z-1)^2} dz^2.$$

Compare [SG, §14.5]. The monodromy of the differential equation $g'' + (1/2)\phi g = 0$ on X determines an isomorphism $\rho : \pi_1(X) \to PSL_2(\mathbb{Z})$.

40. Weyl's Lemma: A weak solution to $\overline{\partial}\phi = 0$ or $\Delta\phi = 0$ is actually smooth.

Proof: suppose $\phi \in \mathcal{D}(U)$ satisfies $\overline{\partial}\phi = 0$. That is, $\int \phi\overline{\partial} f = 0$ for all $f \in C_0^\infty(U)$. Let ϕ_ϵ, f_ϵ, etc. be obtained by convolution with a circularly symmetric bump function of radius ϵ, so $f_\epsilon \to f$ smoothly as $\epsilon \to 0$.

In brief the proof is (a) ϕ_ϵ is smooth and (b) $\phi_\epsilon = \phi$ by the mean-value principle. To prove (b) more precisely we use the fact that $f_\epsilon = f$ if f is smooth and harmonic, and

$$\int (\phi_\epsilon - \phi) f \quad = \quad \int \phi(f - f_\epsilon),$$

so it suffices to show $f - f_\epsilon = \overline{\partial} h$, where h is compactly supported. To this end solve $\overline{\partial} g = f$ on $\mathbb{C}$ and set $h = g - g_\epsilon$. Then g is harmonic for z outside the support of f, so h is compactly supported (if an ϵ-neighborhood of the support of f lies in U), and $(\overline{\partial} g)_\epsilon = (\overline{\partial} g)_\epsilon$, so we are done.

41. Hodge theory on a compact Riemann surface X:

    (a) $H^{0,1}(X)^* = \Omega(X)$ (Dolbeault cohomology classes are dual to holomorphic 1-forms);

    (b) $\alpha = d * df \iff \int_X \alpha = 0$ (a volume form is in the image of the Laplacian iff its integral is zero); and

(c) $H^1_{\mathrm{DR}}(X) = \mathcal{H}^1_\Delta(X)$ (de Rham 1-forms are represented by harmonic 1-forms).

42. Proof of (41c): if ω is a closed 1-form then to make it harmonic we need to solve $d*(\omega+df) = 0 = d*df + d*\omega$, and this is possible since $\int d*\omega = 0$.

43. Proof of (41a): we have $H^{0,1} = \mathcal{A}^1/\overline{\partial}\mathcal{A}^0$. If we can show $\overline{\partial}\mathcal{A}^0$ is closed in the C^∞ topology then we are done, since then $(H^{0,1})^* = (\overline{\partial}\mathcal{A}^0)^\perp = \Omega(X)$ by Weyl's lemma (a distributional 1-form which is perpendicular to the image of $\overline{\partial}$ is holomorphic.)

    We claim $\|df\|_{C^r} \le M_r\|\overline{\partial}f\|_{C^{r+1}}$. If not, there exist smooth functions f_i with $\|df_i\|_{C^r} = 1$ and $\overline{\partial}f_i \to 0$ in C^{r+1}. We can also normalize so $f_i(p) = 0$ at some basepoint, so then f_i is uniformly bounded. Locally we can write
$$f_i = (\overline{\partial}f_i) * \frac{1}{z} + h_i,$$
    where h_i is holomorphic; by assumption the convolution term tends to zero in the C^{r+1} sense, and the h_i are bounded, so we can extract a C^{r+1} convergent subsequence (using the fact that bounded holomorphic functions have smoothly convergent subsequences). Then in the limit, $\overline{\partial}f = 0$ but $\|df\| = 1$, contradicting the fact that f is constant.

    From this inequality we see that if $\omega_i \to \omega$ in C^∞ and $\omega_i = \overline{\partial}f_i$, then $\|df_i\|_{C^r}$ is bounded for every r, so (after normalizing so $f(p) = 0$) there is a convergent subsequence, and the limit f satisfies $\overline{\partial}f = \omega$.

44. Proof of (41b): The proof for the Laplacian is similar: $\Delta(\mathcal{A}^0)$ is smooth closed, and the quotient $\mathcal{A}^2/\Delta(\mathcal{A}^0)$ can be identified with the space of weakly harmonic distributions. But these are just constants, so the image is the forms of integral zero.

45. Dolbeault cohomology groups. In general $H^{p,q}_{\overline{\partial}}(X)$ is the space of $\overline{\partial}$-closed (p,q)-forms, modulo those of the form $\overline{\partial}f$, where f is a $(p,q-1)$ form. For example on a Riemann surface $H^{1,0}_{\overline{\partial}}(X) = \Omega(X)$. One also defines the de Rham group $H^{p,q}$ as the closed (p,q)forms modulo exact ones.

    The *Hodge decomposition* for a compact Kähler manifold states:
$$H^n(X) \cong \oplus H^{p,q}(X) \cong \oplus H^{p,q}_{\overline{\partial}}(X).$$

46. Vanishing theorems: $H^{0,1}_{\overline{\partial}}(\Delta) = 0$. The proof is to write a general $(1,0)$-form ω on the disk as $\omega = \sum \omega_i$ where each ω_i has compact support and the supports tend to infinity. Then solve $\overline{\partial} f_i = \omega_i$ as usual. Finally, for any $r < 1$ and i large enough, we have $\overline{\partial} f_i = 0$ on $\Delta(r)$, so we can approximate f_i there by a polynomial P_i. For suitable approximations $f = \sum(f_i - P_i)$ converges uniformly on compact sets, and $\overline{\partial} f = \omega$.

    More generally, $H^{0,1}_{\overline{\partial}}(X) = 0$ on any noncompact Riemann surface X.

47. Sheaf cohomology $H^p(X, \mathcal{F}) = \lim H^p(X, \mathcal{F}; \mathcal{U})$. We have $H^p(X, \mathcal{F}; \mathcal{U}) \hookrightarrow H^p(X, \mathcal{F})$ for any covering $\mathcal{U}$.

48. Examples:

    (a) $H^0(X, \mathcal{F}) = \mathcal{F}(X)$;

    (b) $H^p(X, \mathcal{F}) = 0$ if $p > \dim X$ (using a covering with all $n+2$-fold intersections equal to zero).

    (c) $H^p(X, \mathcal{E}) = 0$ for $p > 0$, where $\mathcal{E}$ is the sheaf of smooth functions. For $p = 1$: if $g_{ij} \in \mathcal{E}(U_i \cap U_j)$ is given, and U_i is a locally finite covering with a subordinate partition of unity ρ_i, then $f_j = \sum \rho_i g_i j$ defines an element of $\mathcal{E}(U_j)$ (note that $f_{ii} = 0$), and

    $$f_j - f_k = \sum \rho_i (g_{ij} - g_{ik}) = \sum \rho_i g_{kj} = g_{kj}.$$

49. A vanishing theorem: $H^1(\Delta, \mathcal{O}) = 0$. Proof: Given a cocycle $f_{ij} \in \mathcal{O}(U_i \cap U_j)$, first write $f_{ij} = g_j - g_i$ with $\langle g_i \rangle$ smooth. Then $\overline{\partial} g_i = \omega$ gives a global $(0,1)$-form on the disk; solve $\overline{\partial} g = \omega$, replace g_i with $h_i = g_i - g$, and we have $h_i \in \mathcal{O}(U_i)$ with $h_j - h_i = g_{ij}$.

50. The long exact sequence. A short sequence of sheaves

    $$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

    is *exact* if it is exact on the level of stalks. From such a sequence we get a long exact sequence on cohomology,

    $$0 \to H^0(\mathcal{A}) \to H^0(\mathcal{B}) \to H^0(\mathcal{C}) \to H^1(\mathcal{A}) \to H^1(\mathcal{B}) \to \ldots$$

    For example,

    $$0 \to \mathcal{O} \to \mathcal{E} \xrightarrow{\overline{\partial}} \mathcal{E}^{(0,1)} \to 0$$

is exact, and we find quite generally that

$$H^1(X, \mathcal{O}) \cong H^{0,1}_{\overline{\partial}}(X).$$

The special case $X = \Delta$ is what we treated above.

51. The de Rham theorem: $H^p_{\mathrm{DR}}(X) \cong H^p(X, \mathbb{C})$, the latter group being the sheaf or Čech cohomology of X. Proof: from the short exact sequence

$$0 \to \mathcal{Z}^p \to \mathcal{E}^p \xrightarrow{d} \mathcal{E}^{p+1} \to 0$$

we deduce $H^p_{\mathrm{DR}}(X) \cong H^1(X, \mathcal{Z}^{p-1})$, and $H^i(X, \mathcal{Z}^j) \cong H^{i+1}(X, \mathcal{Z}^{j-1})$; so we can get down to $H^p(X, \mathcal{Z}^0)$, and $\mathcal{Z}^0 = \mathbb{C}$.

52. Leray's Theorem: $H^p(X, \mathcal{F}) = H^p(X, \mathcal{F}; \mathcal{U})$ if $\mathcal{U} = \langle U_i \rangle$ is an *acyclic* covering, meaning $H^p(U_i) = 0$ for $p \geq 1$.

Example: $\widehat{\mathbb{C}} = U_1 \cup U_2$ where $U_1 = \mathbb{C}, U_2 = \widehat{\mathbb{C}} - \{0\}$ is an acyclic covering for $\mathcal{O}$; from Laurent series we see any $f_{12}(z) \in \mathcal{O}(U_1 \cap U_2 = \mathbb{C}^*)$ is given by $g_1 - g_2$, $g_i \in \mathcal{O}(U_i)$, so $H^1(\widehat{\mathbb{C}}, \mathcal{O}) = 0$.

53. Dolbeault's theorem: $H^1(X, \Omega) \cong H^{1,1}_{\overline{\partial}}(X) \cong \mathbb{C}$ when X is a compact Riemann surface. The first isomorphism comes from the exact sequence

$$0 \to \Omega \to \mathcal{E}^{1,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{1,1} \to 0,$$

and the second from the fact that $\omega = \overline{\partial}\partial f$ iff $\int \omega = 0$.

54. The Riemann-Roch theorem:

$$\dim H^0(X, \mathcal{O}_D) - \dim H^1(X, \mathcal{O}_D) = 1 - g + \deg D.$$

Here D is a divisor and $\mathcal{O}_D$ is the sheaf of meromorphic functions such that $(f) + D \geq 0$. Proof: (i) true for $D = 0$. (ii) The left-hand side is the same as

$$\chi(\mathcal{O}_D) = \sum (-1)^p \dim H^p(X, \mathcal{O}_d)$$

(since the exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{E}_D \xrightarrow{\overline{\partial}} \mathcal{E}^{0,1}_D \to 0$$

shows $H^p(X, \mathcal{O}_D) = 0$ for $p \geq 2$). (iii) As a general principle, if

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

is exact, then $\chi(\mathcal{B}) = \chi(A) + \chi(C)$, as can be seen from the long exact sequence and the fact that exactness of

$$0 \to V_1 \to V_2 \to V_3 \to \cdots$$

implies $\sum(-1)^i \dim V_i = 0$. (iv) Now from the exact sequence

$$0 \to \mathcal{O}_D \to \mathcal{O}_{D+P} \to \mathbb{C}_P \to 0,$$

where $\mathbb{C}_P$ is the skyscraper sheaf at P, and the easy fact $\chi(\mathbb{C}_P) = 1$, we see Riemann-Roch holds for D iff it holds for $D + P$. (v) Since Riemann-Roch is true for $D = 0$, we have it now for any divisor.

Remark: we do not actually need to know the vanishing of $H^p(X, \mathcal{O})$ for $p > 1$ to prove Riemann-Roch, because $H^1(X, \mathbb{C}_p) = 0$.

55. The Riemann existence theorem: X admits a nonconstant holomorphic map $f : X \to \widehat{\mathbb{C}}$ with $\deg(f) \leq g + 1$. Corollary: X has genus 0 iff $X \cong \widehat{\mathbb{C}}$.

56. Canonical divisors $K = (\omega)$; by Riemann-Roch, $\deg(K) = 2g - 2$. Examples on hyperelliptic surfaces. Alternative proof of Riemann-Hurwitz by pulling back a meromorphic 1-form.

57. Serre Duality: for X a compact Riemann surface,

$$H^1(X, \mathcal{O}_D) \cong H^0(X, \Omega_{-D})^*.$$

We have already seen the special case $H^1(X, \mathcal{O}) \cong \Omega(X)$; the general case follows in exactly the same way, using distributions and duality.

58. Riemann-Roch, restatement:

$$\dim H^0(X, \mathcal{O}_D) = 1 - g + \deg D + i(D),$$

where

$$i(D) = \dim H^0(\mathcal{O}K - D) \geq 0$$

is the *index of speciality* of the divisor D.

59. Examples of Riemann Roch, just using the fact that $i(0) = g$, $i(K) = 1$ and $i(D) = 0$ if $D > K$.

Genus 0:

$$\dim \mathcal{O}_{nK}(X) = \begin{cases} 0 & \text{for } n > 0, \\ 1 & \text{for } n = 0, \text{ and} \\ 1 - 2n & \text{for } n < 0. \end{cases}$$

Note $\dim \mathcal{O}_{-K}(X) = 3 =$ the dimension of $\mathrm{Aut}(X)$.

Genus 1:

$$\dim \mathcal{O}_{nK}(X) = 1$$

for all n since $K = 0$. Hence $\dim \mathrm{Aut}(X) = 1$.

Genus 2:

$$\dim \mathcal{O}_{nK}(X) = \begin{cases} 0 & \text{for } n < 0, \\ 1 & \text{for } n = 0, \\ g & \text{for } n = 1, \\ (2n-1)(g-1) & \text{for } n > 1. \end{cases}$$

60. More on genus 1: since $K = 0$ we find $\dim H^0(X, \mathcal{O}_D) = \deg D$ for any divisor of degree $D > 1$. In particular $\dim H^0(X, \mathcal{O}_{nP}) = 0, 1$ or n according to whether $n < 0$, $0 \leq n \leq 1$ or $n \geq 2$.

61. Hyperelliptic surfaces. A Riemann surface is *hyperelliptic* if it admits a proper map of degree 2 to $\mathbb{P}^1$. We have seen that hyperelliptic surfaces of all genus exist, and that any surface of genus 0 or 1 is hyperelliptic.

62. Theorem: Every surface of genus 2 is hyperelliptic. Proof: consider the map defined by ω_2/ω_1 where ω_1 and ω_2 give a basis for $\Omega(X)$.

    Thus a surface of genus 2 admits a canonical *hyperelliptic involution.*

    Corollaries: $2P$ is a canonical divisor iff P is a critical point of the hyperelliptic map $f : X \to \mathbb{P}^1$. These point can also be characterized by $i(2P) = 1$; they are called the *Weierstrass points* of X. (In particular X is not homogeneous!)

    Note that $i(2P) = 1$ iff there is a meromorphic function with a double pole at P and holomorphic elsewhere.

63. The *canonical map* $f : X \to \mathbb{P}\Omega(X)^*$ can be defined by sending x to the point-evaluation of ω at x; the latter linear functional on $\Omega(X)$ depends on a choice of coordinate at x, but only up to scale. Alternatively, $f(x)$ is the *hyperplane* in $\Omega(X)$ of 1-forms vanishing at X. In coordinates, $f(x) = [\omega_1(x), \ldots, \omega_g(x)]$.

    Note: for genus 2 it is easy to see for all x there is an ω with $\omega(x) \neq 0$; else we can choose a basis vanishing at x, and then ω_1/ω_2 is a degree one map to $\mathbb{P}^1$.

64. Theorem: For a surface of genus two, the canonical map is the unique degree two map $X \to \mathbb{P}^1$.

    Proof: If f is a degree two meromorphic function, then it has a branch point, so we can assume it has a double order pole at some $P \in X$. Then $i(2P) = 1$ so there is an $\omega_1 \in \Omega(X)$ with a double order zero at P. Then $\omega_2 = f\omega_1$ is holomorphic, and $f = \omega_2/\omega_1$ is a canonical map.

65. Cor: The hyperelliptic involution I is central in $\mathrm{Aut}(X)$ for genus 2. In fact I is central in $\mathrm{Mod}(\Sigma_2)$; compare this to the center of $SL_2(\mathbb{Z}) = \mathrm{Mod}(\Sigma_1)$.

66. Moduli space $\mathcal{M}_g$ for $g = 0, 1, 2$. We have

$$\mathcal{M}_0 = \text{a single point},$$

$$\mathcal{M}_1 = \mathbb{P}^1 - \{0, 1, \infty\}/S_4,$$

    and

$$\mathcal{M}_2 = ((\mathbb{P}^1)^3 - \Delta)/S_6,$$

    where the 'discriminant' Δ consists of (x_1, x_2, x_3) such that $x_i = 0$, 1 or ∞, or $x_i = x_j$ for some $i \neq j$.

67. Kodaira-Spencer: the space of infinitesimal deformations of complex structures on X is $H^1(X, \Theta)$, where Θ is the sheaf of holomorphic vector fields on X. For a Riemann surface, $\Theta \cong \mathcal{O}_{-K}$, and thus

$$H^1(X, \Theta) \cong H^1(X, \mathcal{O}_{-K}) \cong H^0(X, \mathcal{O}_{2K}) \cong Q(X),$$

    the space of *holomorphic quadratic differentials* $\phi(z)dz^2$ on X. Note that for $g \geq 2$,

$$\dim \mathcal{M}_g = \dim H^0(X, \mathcal{O}_{2K}) = 3g - 3,$$

    in agreement with the explict case of $g = 2$ above.

68. The Mittag-Leffler problem: constructing a meromorphic function with prescribed principal parts.

On $\mathbb{C}$ this can always be done, for any sequence of principal parts $L_n(z)$ specified at points $p_n \to \infty$. The solution can be given in the form

$$f(z) = \sum L_n(z) + P_n(z),$$

where $P_n(z)$ is a polynomial chosen so that $|L_n(z) - P_n(z)| < 2^{-n}$ on K_n, where $K_1 \subset K_2 \subset \cdots$ is an exhaustion of $\mathbb{C}$ by compact sets.

69. On a compact Riemann surface the principal parts are a global section of $\mathcal{M}/\mathcal{O}$. We have

$$H^0(\mathcal{M}) \to H^0(\mathcal{M}/\mathcal{O}) \to H^1(\mathcal{O}) \cong \Omega(X)^*,$$

and the obstruction to obtaining given data $f = (f_i, U_i)$ is exactly that

$$\sum_X \mathrm{Res}_P(f\omega) = 0$$

for all $\omega \in \Omega(X)$.

To see this, note that $g_{ij} = f_i - f_j = \delta f \in H^1(X, \mathcal{O})$ is corresponds to the cohomology class of $\eta = (\overline{\partial} f_i)$ in $H^{0,1}_{\overline{\partial}}(X)$, and since $\overline{\partial}(1/z)$ is the delta function the residue formula above gives the pairing $\langle \eta, \omega \rangle$.

70. Mittag-Leffler and Riemann-Roch. Now consider the problem of finding all functions f in $H^0(\mathcal{O}_{nP})$, $n \geq 0$. Such an f is determined up to an additive constant by its principal part at P, namely

$$f(z) = \frac{a_n}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \cdots + \frac{a_1}{z} + \text{something holomorphic.}$$

Given a holomorphic 1-form expressed near P as

$$\omega(z) = (b_0 + b_1 z + \cdots b_{n-1} z^{n-1} + O(z^n))dz,$$

we see that

$$\mathrm{Res}_P(f\omega) = b_0 a_1 + \cdots + b_{n-1} a_n.$$

Thus the vector (b_i) determines a linear constraint on the vector (a_i); the 1-forms giving no constraint have a zero of order at least n at P, and thus the number of constraints is

$$\dim \Omega(X) - \dim \Omega_{-nP}(X).$$

Therefore for $D = nP$ the dimension of the solution space is:

$$\begin{aligned}\dim H^0(\mathcal{O}_{nP}) &= \dim(\text{constants} + \text{Laurent tails} - \text{constraints}) \\ &= 1 + n - g + \dim \Omega_{-nP}(X) \\ &= 1 - g + \deg D + i(D),\end{aligned}$$

in agreement with Riemann-Roch.

71. Vector bundles, line bundles and divisors. A holomorphic vector bundle $E \to X$ of rank n is a smooth bundle of complex vector spaces that is locally biholomorphic to $U \times \mathbb{C}^n$ in such a way that the linear structure on the fibers is preserved. Thus E can be specified by a covering U_i and charts

$$E|U_i \xrightarrow{\phi_i} U_i \times \mathbb{C}^n.$$

The *transitions functions*

$$g_{ij} = \phi_i \circ \phi_j^{-1} : U_i \cap U_j \to GL_n(\mathbb{C})$$

(where composition is taken in the fibers) satisfy $g_{ij} g_{jk} = g_{ik}$ and $g_{ii} = \text{id}$. Conversely holomorphic maps g_{ij} of this type determine gluing equations for a vector bundle.

A *section* of E is a map $s : X \to E$; it can be given in local coordinates by maps

$$s_i : U_i \to \mathbb{C}^n$$

satisfying

$$s_i = g_{ij} s_j.$$

72. Theorem: If $E \to X$ is a holomorphic vector bundle over a compact complex manifold, then the space of holomorphic sections $s : X \to E$ is finite-dimensional.

Proof: Use charts or a metric on E to measure the size $\|s\|$ of a section s. Since bounded holomorphic functions have bounded derivatives, the unit ball in this norm is compact. Therefore the space of sections is finite-dimensional.

73. Operations on vector bundles: the transition functions for E^*, $E \oplus F$, $\wedge^n E$ are given by $(g_{ij}^{-1})^t$, $g_{ij} \oplus h_{ij}$ and $\det g_{ij}$. These functors preserve the holomorphic structure. One also has the complex conjugate bundle $\overline{E}$ corresponding to $\overline{g_{ij}}$.

74. Examples: for any complex manifold, the *holomorphic tangent bundle* $T'X \subset TX \otimes \mathbb{C}$ is a holomorphic vector bundle (unlike $TX \otimes \mathbb{C}$, which is only a smooth bundle of complex vector spaces). We have $(T'X)^* = T^{1,0}X$ and $\overline{T'X} = T''X = (T^{1,0}X)^*$.

75. The *canonical bundle* K_X is by definition $K = \bigwedge^n (T'X)^*$, the top exterior power of the holomorphic cotangent bundle. Its holomorphic sections are locally given by $\omega(z)\, dz^1 \wedge \cdots \wedge dz^n$.

76. Line bundles. We let $\mathrm{Pic}(X)$ denote the group of holomorphic line bundles $\mathcal{L} \to X$, up to isomorphism over X, with tensor product as the group operation. Note that $\mathcal{L} \otimes \mathcal{L}^* \cong \mathbb{C}$.

    Theorem: $\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*)$.

77. Line bundles and divisors. We now show (a divisor D on X) corresponds bijectively to (a line bundle $\mathcal{L}_D \to X$ plus a meromorphic section s such that $(s) = D$). The section s is determined by D up to multiplication by an element of $\mathcal{O}^*(X)$.

    In one direction: a divisor D is given locally by $D|U_i = (s_i)$, where s_i is a meromorphic function; then $\mathcal{L}_D$ is determined by $g_{ij} = s_i/s_j$, and $s = (s_i)$ is a meromorphic section of $\mathcal{L}_D$.

    Conversely, if s is a section of $\mathcal{L}$, then $D = (s)$ determines both $\mathcal{L}$ and s up to multiplication by an invertible holomorphic function.

    In terms of sheaf cohomology, the map $D \mapsto \mathcal{L}_D$ is deduced from the exact sequence
    $$0 \to \mathcal{O}^* \to \mathcal{M}^* \to \mathcal{M}^*/\mathcal{O}^*;$$
    the connecting homomorphism yields:
    $$\mathrm{Div}(X) = H^0(\mathcal{M}^*/\mathcal{O}^*) \to H^1(\mathcal{O}^*) \cong \mathrm{Pic}(X).$$

78. Line bundles on $\mathbb{P}^n$. The tautological bundle $\tau \to \mathbb{P}^n$ is defined so $\tau_\ell = \ell$, where ℓ is a line in $\mathbb{C}^{n+1}$. The total space of τ is isomorphic to $\mathbb{C}^{n+1}$ with the origin blown up.

    Theorem: The only global holomorphic section of τ is the zero section, if $n > 0$.

    Proof: such a section gives a holomorphic map $\mathbb{P}^n \to \mathbb{C}^{n+1}$ mapping ℓ into ℓ; this map must be constant, so it takes values in the intersection of the lines ℓ which is the origin.

79. Definition: $\mathcal{O}(d) = (\tau^*)^d$.

Theorem: $H^0(\mathbb{P}^n, \mathcal{O}(d))$ is canonically isomorphic to the space of homogeneous polynomials of degree d on $\mathbb{C}^{n+1}$.

Proof: a section of $\mathcal{O}(d)$ give an element $s(\ell) \in (\ell^*)^d$ for each $\ell \in \mathbb{P}^n$. Thus we can define a holomorphic map $f : \mathbb{C}^{n+1} - \{0\} \to \mathbb{C}$ by setting

$$f(v) = s(\ell)(v)$$

for $v \in \ell$. By Hartog's, f is analytic at the origin. Since $f(\lambda v) = \lambda^d v$, by expanding f in a power series $v = 0$ we see it is a homogeneous polynomial.

Cor: $\mathcal{O}(d)$ corresponds to the divisor dH, where H is a hyperplane.

80. Theorem: $K_{\mathbb{P}^n} \cong \mathcal{L}_{-(n+1)H}$.

Proof: Consider the form $\omega = dz_1 \wedge \dots dz_n$ on $\mathbb{C}^n$. Changing coordinates to another affine chart $(w_1, \dots, w_n)$, we have $z_1 = 1/w_1$, $z_i = w_i/z_1$, and thus

$$\begin{aligned} \omega &= \frac{-dw_1}{w_1^2} \wedge \left(\frac{dw_2}{w_1} - \frac{w_2\, dw_1}{w_1^2} \right) \wedge \cdots \left(\frac{dw_n}{w_1} - \frac{w_n\, dw_1}{w_1^2} \right) \\ &= \frac{-dw_1 \wedge \cdots \wedge dw_n}{w_1^{n+1}}. \end{aligned}$$

This shows $(\omega) = -(n+1)H$.

81. Linear systems. The space $\mathbb{P}H^0(\mathcal{O}_D)$ is the same as the projective space of global holomorphic section of $\mathcal{L}_D$; each such section s is determined by its divisor (s), which is effective since s is holomorphic; the set of all such divisors is denoted $|D|$. It is the *complete linear system* associated to D, and it can be described directly as the space of effective divisors linearly equivalent to D.

A linear system is a linear subvariety $E \subset |D|$. Here E is isomorphic to a projective space $\mathbb{P}^k \cong \mathbb{P}E'$, where E' is a $(k+1)$ dimensional linear space of global holomorphic sections of $\mathcal{L}_D$.

82. The *base locus* B of $E = (E_\lambda)_{\lambda \in \mathbb{P}^k}$ is the set of points common to all the divisors; equivalently $B = E_0 \cap E_1 \cap \dots \cap E_k$, where E_k are divisors corresponding to points in general position in $\mathbb{P}^k$.

If $B = \emptyset$ we say E is *basepoint free.*

83. Theorem: On a compact complex manifold, there is a canonical correspondence:

$$\text{nondegenerate holomorphic maps } f : X \to \mathbb{P}^n, \text{ up to } \mathrm{Aut}(\mathbb{P}^n)$$
$$\Longleftrightarrow$$
$$\text{basepoint-free linear systems } E \text{ on } X.$$

Proof: To f we associate the linear system $f^{-1}(H)$, where H ranges over all hyperplanes in $\mathbb{P}^n$. This is basepoint free because given x there is an H disjoint from $f(x)$; and $f^{-1}(H)$ is a divisor because f is nondegenerate.

Conversely, to a linear system we associate the map $X \to \mathbb{P}H^0(\mathcal{O}_D)^*$, where D is a divisor in E; this map sends x to the linear space of divisors in E passing through x.

Equivalently x corresponds to the subspace of sections vanishing at x; it has codimension 1 because E is basepoint free. With respect to a basis $\langle e_0, \ldots, e_n \rangle$ of the space of sections of $\mathcal{L}_D$ corresponding to E, we can write $f : X \to \mathbb{P}^n$ as $f(x) = [e_0(x) : \cdots : e_n(x)]$.

84. Examples.

(a) On a Riemann surface X, $|K_X|$ is the system of all canonical divisors. For a surface of genus two these are just the fibers of the canonical map $X \to \mathbb{P}^1$.

(b) On $\mathbb{P}^n$, $|dH|$ is the linear system of all hypersurfaces of degree d. For example $|H| \cong (\mathbb{P}^n)^*$ is the space of hyperplanes.

(c) On $X = \mathbb{P}^1$, $|2H|$ maps X to a conic in the plane; $|3H|$ to the twisted cubic in $\mathbb{P}^3$; $|nH|$ to the rational normal curve in $\mathbb{P}^n$.

(d) On $\mathbb{P}^2$, $|2H|$ is the family of conics; it gives the Veronese surface $\mathbb{P}^2 \to \mathbb{P}^5$ so that every conic becomes a hyperplane section.

(Compare Euclid and the theory of 'conic sections'!)

85. Degree and first Chern class. From the exponential sequence we get a natural map

$$\mathrm{Pic}(X) \cong H^1(X, \mathcal{O}^*) \to H^2(X, \mathbb{Z});$$

it sends a line bundle $\mathcal{L}$ to its *first Chern class* $c_1(\mathcal{L})$. For a Riemann surface, $c_1(\mathcal{L}_D) = \deg D$. In general $c_1(\mathcal{L}_D) = [D]$, the cohomology class represented by the zeros and poles of any section.

The degree of a k-dimensional subvariety V in $\mathbb{P}^n$ is the intersection number $V \cdot H^{n-k}$ with a hyperplane of complementary dimension. For a curve this is just $V \cdot H$. If the embedding of V is associated to the divisor D, then the degree is just $[D]^k$ in cohomology. This can be computed as $\int_V c_1(\mathcal{L}_D)^k$.

86. From the exponential sequence we also have an isomorphism between $\mathrm{Pic}_0(X)$, the divisors of degree zero, and $H^1(X, \mathcal{O})$. On a Kähler manifold, $H^1(X, \mathcal{O}) \cong H^{1,0}(X) \subset H^1(X, \mathbb{C})$, so when X is a simply-connected Kähler manifold, line bundles are classified by their first Chern class.

    In particular $\mathrm{Pic}(\mathbb{P}^n) \cong \mathbb{Z}$.

87. Adjunction formula. On a divisor $X \subset Y$ we have 4 line bundles: K_X, K_Y, $\mathcal{L}_X$, and $\mathcal{N}_X = TY/TX$. What relations do they satisfy? From the exact sequence

$$0 \to T^*X \to T^*Y \to \mathcal{N}_X^* \to 0$$

    we get $K_Y \cong K_X \otimes \mathcal{N}_X^*$. Now a function f vanishing on X is determined by its differential df, which is a section of $TX^\perp \cong \mathcal{N}_X^*$; thus $\mathcal{N}_X^* \cong \mathcal{L}_{-X} \cong \mathcal{L}_X^*$. Summing up we get the adjunction formula:

$$K_X = K_Y \otimes \mathcal{L}_X.$$

88. On the level of divisors, this means $K_X = (K_Y + X)|X$. Example: for $X \subset \mathbb{P}^2$ a curve of degree d, we get $K_X = -3H + X$, so

$$2g(X) - 2 = \deg(K_X) = X \cdot (-3H + X) = -3d + d^2,$$

    which shows $g(X) = (d-1)(d-2)/2$.

89. Embeddings of a compact Riemann surface X in projective space.

    $H^0(X, \mathcal{O}_D)$ is basepoint-free iff for all $P \in X$,

$$\dim H^0(O_{D-P}) = \dim H^0(\mathcal{O}_D) - 1.$$

(Equivalently, $\mathcal{O}_D$ is generated by global sections.) $H^0(\mathcal{O}_D)$ gives an embedding iff for all $P, Q \in X$,

$$\dim H^0(O_{D-P-Q}) = \dim H^0(\mathcal{O}_D) - 2.$$

(The case $P = Q$ insures the differential of the map to $\mathbb{P}^n$ is an isomorphism at P).

90. The canonical curve. Theorem: For $g \geq 1$, the complete linear system $|K|$ either gives an embedding $X \mapsto \mathbb{P}^{g-1}$ as a curve of degree $2g-2$, or X is hyperelliptic and the canonical map sends X to the rational normal curve of degree $g-1$ in $\mathbb{P}^{g-1}$.

    Proof: by Riemann-Roch,

$$\dim |K| - \dim |K - P - Q| = 2 + \dim |\emptyset| - \dim |P + Q|.$$

    Since $\dim |\emptyset| = 0$, we can only fail to have an embedding if there exists a P and Q such that $\dim |P+Q| \geq 1$, in which we have a linear system of degree two and dimension one, so X is hyperelliptic.

91. Genus 3: Every curve of genus 3 is either hyperelliptic or isomorphic to a smooth plane curve of degree 4.

    Conversely, by the adjunction formula, $\mathcal{O}(1)|C \cong K_C$ for any plane curve of degree 4; thus $\mathcal{M}_3 = \mathcal{H}_3 \cup \mathcal{C}_4$. Here:

    $\mathcal{H}_3$ is the moduli space of hyperelliptic curves of degree 3, isomorphic to 8-tuples of points on $\mathbb{P}^1$. We have

$$\dim \mathcal{H}_3 = 8 - 3 = 5$$

    since $3 = \dim PGL_2(\mathbb{C})$.

    $\mathcal{C}_4$ is the moduli space of smooth plane curves of degree 4; we have

$$\dim \mathcal{C}_d = \binom{d+2}{2} - 8$$

    since $8 = \dim PGL_3(\mathbb{C})$. In particular $\dim \mathcal{C}_4 = 6$ so the generic genus 3 surface is not hyperelliptic.

92. Geometric interpretation of Riemann-Roch: if D is an effective divisor, then $i(D) = \dim H^0(X, \mathcal{O}_{K-D})$ measures the number of holomorphic 1-forms vanishing on D; but these are the same as hyperplanes in $\mathbb{P}^{g-1}$ passing through the points D under the canonical embedding. Thus if the linear span of D is a $\mathbb{P}^k$, then $i(D) = g - 1 - k$ (its codimension); and Riemann-Roch becomes:

$$\dim H^0(\mathcal{O}_D) = \deg D - \dim \operatorname{span} D.$$

Example: on a smooth plane quartic curve X, if $D = P + Q + R$ lies on a line L, then its span is 1-dimensional, so there is a meromorphic function of degree 3 with polar divisor D. (Since X is neither rational nor hyperelliptic, the degree cannot be 1 or 2.) This function can be given explicitly as projection from S, the remaining point in $L \cap X$.

In particular, $i(3P) = 1$ at the flexes of X, the points where the curvature of X vanishes. These are the Weierstrass points in genus 3.

93. The Jacobian variety. Theorem: integration over 1-cycles determines a map from $H_1(X, \mathbb{Z})$ to a lattice in $\Omega(X)^*$. Proof: if not, then the image of $H^1(X, \mathbb{Z})$ is contained in a real hyperplane; then there is an $\omega \neq 0$ such that $\operatorname{Re} \int_\alpha \omega = 0$ for all cycles α. But $f(q) = \int_p^q \operatorname{Re} \omega$ is a global harmonic function, hence constant, so $\omega = 0$.

We define $\operatorname{Jac}(X) = \Omega(X)^*/H_1(X, \mathbb{Z})$; it is a compact complex manifold of dimension g, homeomorphic to a torus.

94. Theorem (Abel-Jacobi): The space $\operatorname{Div}_0(X)$ of divisors of degree zero, modulo linear equivalence, is canonically isomorphic to $\operatorname{Jac}(X)$, via the map that sends $D = \sum P_i - Q_i$ to the linear functional

$$\phi_D(\omega) = \sum \int_{Q_i}^{P_i} \omega.$$

To make the integral more precise, one chooses a 1-chain α with $\partial \alpha = D$; then $\phi_D(\omega) = \int_\alpha \omega$. Any two choices of α differ by a 1-cycle, so a different choice only changes $\phi_D(\omega)$ by the periods of ω.

The proof of the Abel-Jacobi Theorem comprises 3 steps: (a) showing $\phi_D = 0$ for a principal divisor; (b) showing $\phi_D = 0$ implies $D = (f)$ (Abel's Theorem); and (c) showing the map $\operatorname{Div}_0(X) \to \operatorname{Jac}(X)$ is surjective (Jacobi Inversion).

95. Proof of (a): Suppose $D = (f)$. Choose any smooth path β from 0 to ∞ on $\mathbb{P}^1$, and let $\alpha = f^{-1}(\beta)$. Then $\partial\alpha = D$ and

$$\int_\alpha \omega = \int_\beta f_*\omega = 0$$

since $f_*\omega = 0$, being a holomorphic 1-form on $\mathbb{P}^1$.

96. Abel's Theorem: Suppose $\partial\alpha = D$ and $\int_\alpha \omega = 0$ for all $\omega \in \Omega(X)$. Then there is a meromorphic function h such that $D = (h)$.

    Proof of Abel's Theorem (and Step (b)). Note that the goal can be stated equivalently as saying we want to show $\mathcal{L}_D$ is the trivial bundle, since $\mathcal{L}_D$ somes equipped with a section such that $(s) = D$.

97. The first step is to find a smooth nonvanishing section f of $\mathcal{L}_D$. This means, in terms of divisors, that we find a section of $\mathcal{E}_D^*$, i.e. a function that is locally of the form $g(z)/z^n$, g smooth and nonzero, near P with $D(P) = n$.

    To see f exists, first consider the case of $D = P - Q$ in $\mathbb{C}$; then $f(z) = (z-Q)/(z-P)$ works. Since f tends to 1 at ∞ we can smooth it off to be constant outside a neighborhood of P and Q, then transplant it to X. This shows there exists weak solutions to $(f) = D$ whenever P_i and Q_i are close on X. Now moving P_i in steps to Q_i we get a weak solution for any divisor of degree zero.

    (One can also use the fact that line bundles on a surface are classified topologically by their degree, so $\mathcal{L}_D$ is smoothly trivial).

98. The second step is to note that for any closed 1-form η on X,

$$\int_\alpha \eta = \frac{1}{2\pi i}\int \frac{df}{f} \wedge \eta.$$

    It suffices to check this in the local picture for $D = P - Q$ on $\mathbb{C}$. There df/f is compactly supported, so we can write $\eta = dg$ on the region of integration. Also

$$\frac{df}{f} = \frac{dz}{z-Q} - \frac{dz}{z-P}$$

    near P and Q. We find

$$\frac{1}{2\pi i}\int \frac{df}{f} \wedge dg = -\frac{1}{2\pi i}\int d\left(\frac{df}{f}\right) \wedge g = g(P) - g(Q) = \int_Q^P dg = \int_Q^P \eta,$$

using the fact that $\overline{\partial}(1/z) = -2\pi i \delta(z)$ as a distribution.

99. The final step is to modify f so it becomes holomorphic. To this end note that for a smooth g,

$$\overline{\partial}(e^{-g} f) = e^{-g}(\overline{\partial} f - f \overline{\partial} g).$$

So if we can find g such that $\overline{\partial} g = \overline{\partial} f / f = \eta$, then we have completed the proof, since $h(z) = e^{-g(z)} f(z)$ is then a meromorphic function with divisor D.

But from the isomorphism $H^{0,1}_{\overline{\partial}}(X) \cong \Omega(X)^*$, we know the equation $\overline{\partial} g = \eta$ has a solution if and only if $\int \eta \wedge \omega = 0$ for all $\omega \in \Omega(X)$. Since

$$\int_X \eta \wedge \omega = \int_X \frac{df}{f} \wedge \omega = \int_\alpha \omega = 0$$

by the hypothesis of Abel's theorem, we are done.

100. Theorem (Jacobi Inversion). The map $\mathrm{Div}_0(X) \to \mathrm{Jac}(X)$ is surjective.

Proof of Jacobi Inversion (and Step (c)): Clearly there exist $x_1, \ldots, x_g$ such that $(\omega(x_i) = 0$ for all $i) \implies \omega = 0$. (Choose the x_i by induction and use the fact that $\dim \Omega(X) = g$.)

Fixing any $(Q_1, \ldots, Q_g)$, consider the map $F : X^g \to \mathrm{Jac}(X)$ given by

$$(P_1, \ldots, P_g) \to \sum \int_{Q_i}^{P_i} \omega.$$

Now the adjoint of the derivative of this map, $(DF)^*$, sends $\Omega(X) \to T^* X^g$; it is given by $(DF)^*(\omega) = \langle \omega(P_i) \rangle$. Thus DF is surjective at $(P_i) = x_i$, since $(DF)^*$ has no kernel. Therefore $F(X^g)$ is open; it is also closed, since X^g is compact, so F is surjective. But F is part of the map on divisors, QED.

101. The case $g = 1$; here $X = \mathbb{C}/\Lambda$. We have $\Omega(X) = \langle dz \rangle$, with periods Λ; thus $\mathrm{Jac}(X) \cong X$ and $D = \sum P_i - Q_i \in \mathrm{Div}(X)$ is principal iff the group-theoretic sum $\sum P_i - Q_i \in X$ vanishes.

Examples: (a) The zeros z_1 and z_2 of $\wp(z)$ satisfy $z_1 = -z_2$. (b) If $D = P - Q$ maps to zero in the Jacobian, then $P = Q$, which implies $D = 0$. Thus there is no degree one map $X \to \mathbb{P}^1$.

102. Sheaf theory approach: from the exponential sequence we get

$$H^1(X,\mathbb{Z}) \to H^1(X,\mathcal{O}) \cong \Omega(X)^* \to H^1(X,\mathcal{O}^*) \to H^2(X,\mathbb{Z}) \cong \mathbb{Z}.$$

Thus $\mathrm{Pic}_0(X)$, the subgroup of $H^1(X,\mathcal{O}^*)$ with vanishing first Chern class, is isomorphic to $\Omega(X)^*/H^1(X,\mathbb{Z})$, and the latter is another description of $\mathrm{Jac}(X)$.

103. Jacobian embedding and the canonical map. Fix $Q \in X$, and consider the derivative of the map $X \to \mathrm{Jac}(X)$ given by $P \mapsto \mathcal{L}_{P-Q}$. The tangent space to any point of $\mathrm{Jac}(X)$ is naturally identified with $\Omega(X)^*$, so we get a map $X \to \mathbb{P}\Omega(X)^*$. This is exactly the canonical map.

104. Geometry of the Jacobian for $g > 1$. For $g = 1$ we have $\mathrm{Jac}(X) \cong \mathbb{C}/(\mathbb{Z} \oplus \tau\mathbb{Z})$, where $\mathrm{Im}\,\tau > 0$. In general $\mathrm{Jac}(X) \cong \mathbb{C}^g/(\mathbb{Z}^g \oplus T\mathbb{Z}^g)$, where T_{ij} is a symmetric matrix and $\mathrm{Im}\,T_{ij}$ is positive-definite. The proof of these facts (Riemann's bilinear relations) relies on Poincaré duality.

105. Poincaré duality. Let $\alpha_i, \beta_i, i = 1, \ldots, g$ be a canonical basis for $H_1(X,\mathbb{Z})$. Then the alternating unimodular intersection pairing on homology is given by $\alpha_i \cdot \beta_i = -\beta_i \cdot \alpha_i = 1$; the other intersections are zero. From this we have Poincaré duality, $H^1(X,\mathbb{C}) \cong H_1(X,\mathbb{C})$.

106. Product on forms. Given elements $\omega, \omega' \in H^1_{\mathrm{DR}}(X,\mathbb{C})$ the alternating pairing

$$(\omega,\omega') = \int_X \omega \wedge \omega'$$

is defined. We claim that via Poincaré duality this pairing on de Rham cycles is the same (up to sign) as the intersection pairing on homology. To see this, we define the Poincaré dual of $A \in H_1(X)$ to be the form A^* such that

$$\int_B A^* \quad = \quad A \cdot B.$$

Let α_i^*, β_i^* be de Rham cycles dual to a canonical basis for H_1. The form α_i^*, for example, is supported on a collar neighborhood of α_i isomorphic to $[0,1] \times \alpha_i$, and it is locally given by the pullback of any bump form ω on $[0,1]$ with $\int \omega = 1$. Then taking into account orientation and supports, we find

$$(\alpha_i^*, \beta_i^*) = \alpha_i \cdot \beta_i = -1$$

and more generally

$$(A^*, B^*) = A \cdot B.$$

107. Product on period matrices. We can also record an element ω in $H^1_{\mathrm{DR}}(X)$ by its periods $\Pi \in H_1(X)^*$, that is by the complex numbers $a_i = \int_{\alpha_i} \omega$, $b_i = \int_{\beta_i} \omega$. Letting Π' denote the periods of ω', and

$$(\Pi, \Pi') = \sum_i a_i b_i' - a_i' b_i,$$

we have $(\Pi, \Pi') = (\omega, \omega')$.

108. Example: On the torus $X = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i$, we have cycles α, β corresponding to the x and y axes; then $\alpha^* = dy$, $\beta^* = -dx$, and we see

$$(\alpha^*, \beta^*) = \int -dy \wedge dx = 1 = \alpha \cdot \beta.$$

Also the periods of α^* on $(\alpha, beta)$

$$\Pi = (0, 1) = (a, b) = (\alpha \cdot \alpha, \alpha \cdot \beta),$$

and

$$\Pi' = (-1, 0) = (a', b') = (\beta \cdot \alpha, \beta \cdot \beta),$$

so we can check that

$$(\Pi, \Pi') = ab' - a'b = 1 = (\alpha^*, \beta^*).$$

109. There is a nice geometric proof of this formula on a torus X; since $g = 1$ we will drop the subscript i. Then cutting along α, β we obtain a square S bounded, in counterclockwise order, by α^+, β^+, α^- and β^-. (The signs indicate the relation between the orientation induced by S and by the cycle on X.) Given ω, ω' we can write $\omega = d\eta$ on S, and then by Stokes' theorem:

$$\int_X \omega \wedge \omega' = \int_{\partial S} \eta \wedge \omega'.$$

Under the natural identification on X, we have $d\eta|\alpha^+ = d\eta|\alpha^-$, so $\eta|\alpha^+ - \eta|\alpha^- = C$. What is this constant C? We can find it by integrating $d\eta$ from a point on α^- to the corresponding point on α^+. But β^- provides such a path, so $C = \int_{\beta^-} d\eta = -\int_\beta \omega = -b$. Therefore

$$\int_{\alpha^+\cup\alpha^-} \eta \wedge \omega' \quad = \quad \int_\alpha -b\omega' \quad = \quad -ba'.$$

Similarly,

$$\eta|\beta^+ - \eta|\beta^- = \int_{\alpha^+} d\eta = a,$$

so

$$\int_{\beta^+\cup\beta^-} \eta \wedge \omega' \quad = \quad \int_\beta a\omega' \quad = \quad ab'.$$

Thus $(\omega, \omega') = ab' - b'a = (\Pi, \Pi')$.

110. On a general surface of genus g, we can choose a canonical basis such that $\alpha_i \cup \beta_i$ and $\alpha_j \cup \beta_j$ are disjoint for $i \neq j$. Then cutting along these curves we obtain a surface with boundary S, such that ω has a primitive η on S. (Even though S is not a disk, all cycles on S are null-homologous on X). Then the argument above can be applied to the g boundary components of S, showing

$$(\omega, \omega') \quad = \quad \sum a_i b_i' - a_i' b_i.$$

111. Riemann's bilinear relations. We claim that there is a unique basis of $\Omega(X)$ such that $\int_{\alpha_i} \omega_j = \delta_{ij}$, and that with respect to this basis, the remaining periods

$$b_{ij} = \int_{\beta_i} \omega_j$$

satisfy (I) $b_{ij} = b_{ji}$ and (II) $\mathrm{Im}(b_{ij}) > 0$. Condition (II) means the real matrix is positive definite.

Note that (II) generalizes the fact that $\tau \in \mathbb{H}$ for $g = 1$.

First, to prove there exists a normalized basis, observe that if $a_i = 0$ for an $\omega \in \Omega(X)$ then

$$\|\omega\|^2 = \frac{i}{2}\int_X \omega \wedge \bar{\omega} \quad = \quad \sum a_i \bar{b}_i - \bar{a}_i b_i = 0.$$

Thus $\Omega(X)$ is determined by its α-periods, and since $\dim \Omega(X) = g$ there exists a dual basis.

Now for (I) we use the fact that

$$0 = \int \omega_j \cap \omega_k = \sum_i a_{ij} b_{ik} - a_{ik} b_{ij} = b_{jk} - b_{kj}.$$

For (II) we use the fact that for $c_j \in \mathbb{R}$, not all zero, we have

$$\begin{aligned} 0 &< \|\sum c_j \omega_j\|^2 = \frac{i}{2} \int \sum_{j,k} c_j c_k \omega_j \wedge \bar{\omega}_k \\ &= \frac{i}{2} \sum_{i,j,k} c_j c_k (a_{ij} \bar{b}_{ik} - \bar{a}_{ik} b_{ij}) \\ &= \sum_{j,k} \frac{i}{2} (\bar{b}_{jk} - b_{kj}) c_j c_k \\ &= \sum_{j,k} (\operatorname{Im} b_{jk}) c_j c_k, \end{aligned}$$

which says exactly that $\operatorname{Im} b_{jk}$ is positive definite.

112. Theorem (Belyi). A Riemann surface X is defined over a number field iff it admits a map to $\mathbb{P}^1$ branched over no more than 3 points.

Proof, in one direction. Suppose X is defined over a number field. Then it is branched over a finite set B of algebraic points. We then find a composition of polynomial maps $P : \mathbb{P}^1 \to \mathbb{P}^1$ such that $P(B) \cup$ (the critical values of P) is $\{0, 1, \infty\}$.

First, if x in B is of degree d over the rationals, then we can find a rational polynomial P of degree d vanishing at x. The critical values of P have degree at most $d-1$ over the rationals, so we can eventually arrange that B lies in $\mathbb{Q}$.

Second, consider the polynomial $P(z) = az^n(1-z)^m$. This has a critical points only at 0, 1, ∞ and $n/(m+n)$, and by choosing a appropriately we can arrange that the critical values are 0, 1 and ∞. Thus P reduces the number of points in B by one, by removing the critical value at $n/(m+n)$. Eventually we reduce to $B = \{0, 1, \infty\}$.

Proof, the other direction. X and its branched cover can be defined over some finitely generated extension of $\mathbb{Q}$. Think of the transcendentals in this extension as variables parameterizing a family of Riemann surfaces. Since the triply punctured sphere is rigid, the isomorphism type of X is constant in any component of this family. So by specializing these values to algebraic numbers, we obtain a curve defined over a number field.

113. Remark: Belyi's theorem shows that X is defined over a number field iff it can be built out of equilateral triangles. One direction is by pulling back a triangulation of the triply-punctured sphere; the other is by making a barycentric subdivision if necessary, then "folding up" to get $\mathbb{P}^1$.

References

[SG] G. Sansone and J. Gerretsen. *Lectures on the Theory of Functions of a Complex Variable*. P. Noordhoff, 1960.