Course Outline

Complex Analysis

Math 205, Spring 1995, Berkeley CA

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Texts:

Nehari, *Conformal Mapping*

Ahlfors, *Lectures on Quasiconformal Mappings.*

Also recommended:

Ahlfors, *Complex Analysis*

Conway, *Functions of One Complex Variable*

Lehto, *Univalent Functions and Teichmüller Theory*

1. Background in real analysis and basic differential topology (such as covering spaces and differential forms) is a prerequisite.

2. Relations of complex analysis to other fields include: algebraic geometry, complex manifolds, several complex variables, Lie groups and homogeneous spaces ($\mathbb{C}, \mathbb{H}, \widehat{\mathbb{C}}$), geometry (Platonic solids; hyperbolic geometry in dimensions two and three), Teichmüller theory, elliptic curves and algebraic number theory, $\zeta(s)$ and prime numbers, dynamics (iterated rational maps).

3. Algebraic origins of complex analysis; solving cubic equations $x^3 + ax + b = 0$ by Tchirnhaus transformation to make $a = 0$. This is done by introducing a new variable $y = cx^2 + d$ such that $\sum y_i = \sum y_i^2 = 0$; even when a and b are real, it may be necessary to choose c complex (the discriminant of the equation for c is $27b^2 + 4a^3$.) It is negative when the cubic has only one real root; this can be checked by looking at the product of the values of the cubic at its max and min.

4. Elements of complex analysis: $\mathbb{C} = \mathbb{R}[i]$, $\overline{z}$ (and related examples for $\mathbb{Q}[\sqrt{2}]$). Geometry of multiplication: why is it conformal? (Because $|ab| = |a||b|$, so triangles are mapped to similar triangles!) Visualizing $e^z = \lim(1 + z/n)^n$. A pie slice centered at $-n$ and with angle π/n is mapped to the upper semi-circle; in the limit we find $\exp(\pi i) = -1$.

5. Definition: $f(z)$ is analytic if $f'(z)$ exists. Note: we do not require continuity of f'! Cauchy's theorem: $\int_\gamma f(z)dz = 0$. Plausibility: $\int_\gamma z^n dz = \int_a^b \gamma(t)^n \gamma'(t)dt = \int_a^b (\gamma(t)^{n+1})/(n+1)'dt = 0$.

   Proof 1: $f(z)dz$ is a closed form, when f is holomorphic, assuming $f(z)$ is smooth. Discussion: f is holomorphic iff $idf/dx = df/dy$; from this $d(fdz) = 0$. Moreover $df/dx = df/dz$, where $d/dz = 1/2(d/dx + (1/i)d/dy)$. We have f analytic if and only if $df/d\overline{z} = 0$. Then $df = (df/dz)dz + (df/d\overline{z})d\overline{z}$ and we see $d(fdz) = 0$ iff $df/d\overline{z} = 0$ iff f is holomorphic.

   Proof 2: (Goursat), assuming only complex differentiability.

6. Analyticity and power series. The fundamental integral $\int_\gamma dz/z$. The fundamental power series $1/(1-z) = \sum z^n$. Put these together with Cauchy's theorem,
$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{\zeta - z},$$
   to get a power series.

   Theorem: $f(z) = \sum a_n z^n$ has a singularity (where it cannot be analytically continued) on its circle of convergence $|z| = R = 1/\limsup |a_n|^{1/n}$.

7. Infinite products: $\prod(1+a_n)$ converges if $\sum |a_n| < \infty$. The proof is in two steps: first, show that when $\prod(1+|a_n|)$ converges, the differences in successive partial products bound the differences for $\prod(1+a_n)$. Then, show $\sum |a_n| \le \prod(1+|a_n|) \le \exp(\sum |a_n|)$.

   Example: evaluation of $\zeta(2) = \prod(1/(1-1/p^2))$, where p ranges over the primes, converges (to $\pi^2/6$).

8. Cauchy's bound $|f^{(n)}(0)| \le n! M(R)/R^n$. Liouville's theorem; algebraic completeness of $\mathbb{C}$. Compactness of bounded functions in the uniform topology. Parseval's inequality (which implies Cauchy's:)
$$\sum |a_n|^2 R^2 = \frac{1}{2\pi}\int_{|Z|=R} |f(z)|^2 d\theta \le M(R)^2.$$

9. Morera's theorem (converse to Cauchy's theorem). Definition of $\log(z) = \int_1^z d\zeta/\zeta$. Analytic continuation, natural boundaries, $\sum a_n z^{n!}$. Laurent series: $f(z) = \sum_{-\infty}^{\infty} a_n z^n$ where $a_n = (1/2\pi i)\int_C f(z)/z^{n+1}dz$.

Classification of isolated singularities; removability of singularities of bounded functions. Behavior near an essential singularity (Weierstrass-Casorati): $\overline{f(U)} = \mathbb{C}$.

10. Generating functions and $\sum F_n z^n$, F_n the nth Fibonacci number. A power series represents a rational function iff its coefficients satisfiy a recurrence relation. Pisot numbers, the golden ratio, and why are 10:09 and 8:18 such pleasant times.

11. Kronecker's theorem: one need only check that the determinants of the matrices $a_{i,i+j}$, $0 \leq i, j \leq n$ are zero for all n sufficiently large.

12. Residue theorem and evaluation of definite integrals. Three types: $\int_0^{2\pi} R(\cos\theta, \sin\theta) d\theta$, $\int_{-\infty}^{\infty} R(x) dx$, and $\int_0^{\infty} x^a R(x) dx$, $0 < a < 1$, R a rational function.

13. Hardy's paper on $\int \sin(x)/x\, dx$.

14. The argument principle: number of zeros - number of poles is equal to

$$\frac{1}{2\pi i} \int_{\partial\Omega} \frac{f'(\zeta)}{f(\zeta)} d\zeta.$$

    Similary, the weighted sum of $g(z)$ over the zeros and poles is given by multiplying the integrand by $g(\zeta)$.

    Winding numbers of the topological nature of the argument principle: if a continuous $f : \overline{\Delta} \to \mathbb{C}$ has nonzero winding number on the circle, then f has a zero in the disk.

15. Rouché's theorem: if $|g| < |f|$ on $\partial\Omega$, then $f + g$ and g have the same number of zeros-poles in Ω. Example: $z^2 + 15z + 1$ has all zeros of modulus less than 2, but only one of modulus less than 3/2.

16. Open mapping theorem: if f is nonconstant, then it sends open sets to open sets. Cor: the maximum principle ($|f|$ achieves its maximum on the boundary).

17. Invertibility. (a) If $f : U \to V$ is injective and analytic, then f^{-1} is analytic. (b) If $f'(z) \neq 0$ then f is locally injective at z. Formal inversion of power series.

18. Phragmen -Lindelöf type results: if f is bounded in a strip $\{a < \mathrm{Im}(z) < b\}$, and f is continuous on the boundary, then the sup on the boundary is the sup on the interior.

19. Hadamard's 3-circles theorem: if f is analytic in an annulus, then $\log M(r)$ is a convex function of $\log r$, where $M(r)$ is the sup of $|f|$ over $|z| = r$. Proof: a function $\phi(s)$ of one real variable is convex if and only if $\phi(s) + ar$ satisfies the maximum principle for any constant a. This holds for $\log M(\exp(s))$ by considering $f(z)z^a$ locally.

20. The concept of a Riemann surface; the notion of isomorphism; the three simply-connected Riemann surface $\mathbb{C}$, $\widehat{\mathbb{C}}$ and $\mathbb{H}$.

21. A nonconstant map between compact surfaces is surjective, by the open mapping theorem.

22. Theorem: $\mathrm{Aut}(\mathbb{C}) = \{az + b\}$.

23. The complex plane $\mathbb{C}$. The notion of metric $\rho(z)|dz|$. The automorphism group is solvable. Inducing a metric $|dz|/|z|$ on $\mathbb{C}^*$. The cone metric $|dz|/|\sqrt{z}|$ giving the quotient by z^2.

24. The Riemann sphere $\widehat{\mathbb{C}}$ and its automorphisms. Theorem: $\mathrm{Aut}(\widehat{\mathbb{C}}) = PSL_2(\mathbb{C})$. A particularly nice realization of this action is as the projectivization of the linear action on $\mathbb{C}^2$.

25. Möbius transformations: invertible, form a group, act by automorphisms of $\widehat{\mathbb{C}}$, triply-transitive, sends circles to circles. Proof of last: a circle $x^2 + y^2 + Ax + By + C = 0$ is also given by $r^2 + r(A\cos\theta + B\sin\theta) + C = 0$, and it is easy to transform the latter under $z \mapsto 1/z$, which replaces r by $1/\rho$ and θ by $-\alpha$.

26. Classification of Möbius transformations and their trace squared: (a) identity, 4; (b) parabolic (a single fixed point) 4; (c) elliptic (two fixed points, derivative of modulus one) $[0, 4)$; (d) hyperbolic (two fixed points, one attracting and one repelling) $\mathbb{C} - [0, 4]$.

27. Stereographic projection preserves circles and angles. Proof for angles: given an angle on the sphere, construct a pair of circles through the north pole meeting at that angle. These circles meet in the same angle at the pole; on the other hand, each circle is the intersection of the

sphere with a plane. These planes meet $\mathbb{C}$ in the same angle they meet a plane tangent to the sphere at the north pole, QED.

28. Four views of $\widehat{\mathbb{C}}$: the extended complex plane; the Riemann sphere; the Riemann surface obtained by gluing together two disks with $z \mapsto 1/z$; the projective plane for $\mathbb{C}^2$.

29. The spherical metric $2|dz|/(1+|z|^2)$. Derive from the fact "Riemann circle" and the map $x = \tan(\theta/2)$, and conformality of stereographic projection.

30. Some topology of projective spaces: $\mathbb{RP}^2$ is the union of a disk and a Möbius band; the Hopf map $S^3 \to S^2$.

31. Gauss-Bonnet for spherical triangles: area equals angle defect. Prove by looking at the three lunes (of area 4θ) for the three angles of a triangle. General form: $2\pi\chi(X) = \int_X K + \int_{\partial X} k$.

32. Theorem: $\operatorname{Aut}(\mathbb{H}) = PSL_2(\mathbb{R})$. Schwarz Lemma and automorphisms of the disk. The hyperbolic metric $|dz|/\operatorname{Im}(z)$ on $\mathbb{H}$, and its equivalence to $2|dz|/(1-|z|^2)$ on Δ.

33. Classification of automorphisms of $\mathbb{H}^2$, according to translation distance.

34. Hyperbolic geometry: geodesics are circles perpendicular to the circle at infinity. Euclid's fifth postulate (given a line and a point not on the line, there is a unique parallel through the point. Here two lines are parallel if they are disjoint.)

35. Gauss-Bonnet in hyperbolic geometry. (a) Area of an ideal triangle is $\int_{-1}^{1}\int_{\sqrt{1-x^2}}^{\infty}(1/y^2)dydx = \pi$. (b) Area $A(\theta)$ of a triangle with two ideal vertices and one external angle θ is additive ($A(\alpha+\beta) = A(\alpha)+A(\beta)$) as a diagram shows. Thus $A(\alpha) = \alpha$. (c) Finally one can extend the edges of a general triangle T in a spiral fashion to obtain an ideal triangle containing T and 3 other triangles, each with 2 ideal vertices.

36. Harmonic functions. A real-value function $u(z)$ is harmonic iff u is locally the real part of an analytic function; indeed, harmonic means $d(*du) = 0$, and $v = \int *du$. Harmonic functions are preserved under

analytic mappings. Examples: electric potential; fluid flow around a cylinder.

The mean-value principle ($u(0)$ = average over the circle) follows from Cauchy's formula, as does the Poisson integral formula ($u(p)$ = visual average of u).

37. The Schwarz reflection principle: if $U = U^*$, and f is analytic on $U \cap \overline{\mathbb{H}}$, continuous and real on the boundary, then $\overline{f(\overline{z})}$ extends f to all of U. This is easy from Morera's theorem. A better version only requires that $\mathrm{Im}(f) \to 0$ at the real axis, and can be formulated in terms of harmonic functions (cf. Ahlfors):

    If v is harmonic on $U \cap \overline{\mathbb{H}}$ and vanishes on the real axis, then $v(\overline{z}) = -v(z)$ extends v to a harmonic function on U. For the proof, use the Poisson integral to replace v with a harmonic function on any disk centered on the real axis; the result coincides with v on the boundary of the disk and on the diameter (where it vanishes by symmetry), so by the maximum principle it is v.

38. Reflection gives another proof that all automorphisms of the disk extend to the sphere.

39. Normal families: any bounded family of analytic functions is normal, by Arzela-Ascoli.

40. Riemann mapping theorem: given a simply-connected region $U \subset \mathbb{C}$, $U \neq \mathbb{C}$, and a basepoint $u \in U$, there is a unique conformal homeomorphism $f : (U, u) \to (\Delta, 0)$ such that $f'(u) > 0$. Proof: let $\mathcal{F}$ be the family of univalent maps $(U, u) \to (\Delta, 0)$. Using a square-root and an inversion, show $\mathcal{F}$ is nonempty. Also $\mathcal{F}$ is closed under limits. By the Schwarz Lemma, $|f'(u)|$ has a finite maximum over all $f \in \mathcal{F}$. Let f be a maximizing function. If f is not surjective to the disk, then we can apply a suitable composition of a square-root and two automorphisms of the disk to get a $g \in \mathcal{F}$ with $|g'(u)| > |f'(u)|$, again using the Schwarz Lemma. QED.

41. Uniformization of annuli: any doubly-connected region in the sphere is conformal isomorphic to $\mathbb{C}^*$, Δ^* or $A(R) = \{z \,:\, 1 < |z| < R\}$. The map from $\mathbb{H}$ to $A(R)$ is $z \mapsto z^\alpha$, where $\alpha = \log(R)/(\pi i)$. The deck transformation is given by $z \mapsto \lambda z$, where $\lambda = 4\pi^2/\log(R)$.

42. The class S of univalent maps $f : \Delta \to \mathbb{C}$ such that $f(0) = 0$ and $f'(0) = 1$. Compactness of S. The Bieberbach Conjecture/de Brange Theorem: $f(z) = \sum a_n z^n$ with $|a_n| \le n$.

43. The area theorem: if $f(z) = z + \sum b_n/z_n$ is univalent on $\{z : |z| > 1\}$, then $\sum n|b_n|^2 < 1$. The proof is by integrating $\overline{f} df$ over the unit circle and observing that the result is proportional to the area of the complement of the image of f.

44. Proof that $|a_2| \le 2$: first, apply the area theorem to conclude $|a_2^2 - a_3| \le 1$. Then consider $\sqrt{f(z^2)}$ for $f \in S$.

45. The Koebe 1/4 Theorem: if $f \in S$ then $f(\Delta) \supset \Delta(1/4)$. Proof: if w is omitted from the image, then $f(z)/(1 - f(z)/w) \in S$; now apply $|a_2| \le 2$.

46. Riemann surfaces and holomorphic 1-forms. The naturality of the residue, and of df.

47. The Residue Theorem: the sum of the residues of a meromorphic 1-form on a compact Riemann surface is zero. Application to df/f, and thereby to the degree of a meromorphic function.

48. Remarks on 1-forms: a holomorphic 1-form on the sphere is zero, because it integrates to a global analytic function. Moreover a meromorphic 1-form on the sphere always has 2 more poles than zeros.

49. The Schwarz-Christoffel formula. Let $f : \mathbb{H} \to U$ be the Riemann mapping to a polygon with vertices p_i, $i = 1, \ldots, n$ and exterior angles $\pi\mu_i$. Then
$$f(z) = \alpha \int \frac{d\zeta}{\prod_1^n (\zeta - q_i)^{\mu_i}} d\zeta \ + \beta,$$
where $f(q_i) = p_i$.

    Proof: compute the nonlinearity $N(f) = f''(z)/f'(z)dz$. By Schwarz reflection, it extends to a meromorphic 1-form on the sphere with simple poles at the q_i. Using the fact that $z^{(1-\mu_i)}$ straightens out the ith vertex of U, one finds that $N(f)$ has residue $-\mu_i$ at q_i. Since a 1-form is determined by its singularities, we have
$$N(f) = \sum \frac{-\mu_i dz}{(z - q_i)},$$

and the formula results by integration.

50. Examples of Schwarz-Christoffel: $\log(z) = \int d\zeta/\zeta$ (maps to a bigon with external angles of π); $\sin^{-1}(z) = \int d\zeta/\sqrt{1-\zeta^2}$ (maps to a triangle with external angles $\pi/2$, $\pi/2$ and π.)

51. The length-area method. Let $f : R(a,b) \to Q$ be a conformal map of a rectangle to a Jordan region $Q \subset \mathbb{C}$, where $R(a,b) = [0,a] \times [0,b] \subset \mathbb{C}$, Then there is a horizontal line $[0,a] \times \{y\}$ whose image has length $L^2 \le (a/b)\,\text{area}(Q)$. Similarly for vertical lines.

    Corollary: given any quadrilateral Q, the product of the minimum distances between opposite sides is a lower bound for area(Q).

52. Theorem: The Riemann map to a Jordan domain extends to a homeomorphism on the closed disk. Proof: given a point $z \in \partial\Delta$, map Δ to an infinite strip, sending z to one end. Then there is a sequence of disjoint squares in the strip tending towards that end. The images of these squares have areas tending to zero, so there are cross-cuts whose lengths tend to zero as well, by the length-area inequality. This gives continuity at z. Injectivity is by contradiction: if the map is not injective, then it is constant on some interval along the boundary of the disk.

53. Weierstrass/Hadamard factorization theory. (A good reference for this material is Titchmarsh, *Theory of Functions*). We will examine the extent to which an entire function is determined by its zeros. We beginning by showing that any discrete set in $\mathbb{C}$ arises as the zeros of an entire function.

54. Weierstrass factor. Inspired by the fact that $(1-z)\exp\log 1/(1-z) = 1$ and that $\log 1/(1-z) = z + z^2/2 + z^3/3 + \ldots$, we set

$$E_p(z) = (1-z)\exp\left(z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right).$$

    By convention $E_0(z) = (1-z)$.

    Theorem: For $|z| < 1$, we have $|E_p(z) - 1| \le |z|^{p+1}$.

    Proof: Writing $E_p(z) = 1 + \sum a_k z^k$, one may check (by computing $E_p'(z)$) that all $a_k < 0$, $\sum |a_k| = 1$, and $a_1 = a_2 = \ldots = a_p = 0$. Then $|E_p(z) - 1| = |\sum a_k z^k| \le |z|^{p+1} \sum |a_k| \le |z|^{p+1}$. QED

55. Theorem: If $\sum (r/|a_n|)^{p_n+1} < \infty$ for all $r > 0$, then $f(z) = \prod E_{p_n}(z/a_n)$ converges to an entire function with zeros exactly at the a_n.

    Cor: Since $p_n = n$ works for any $a_n \to \infty$, we have shown any discrete set arises as the zeros of an entire function.

56. Blaschke products. Let $f : \Delta \to \Delta$ be a proper map of degree d. Then

$$f(z) = e^{i\theta} \prod_1^d \left(\frac{z - a_i}{1 - \overline{a_i} z} \right)$$

    where the a_i enumerate the zeros of f.

57. Jensen's formula. Let $f(z)$ be holomorphic on the disk of radius R about the origin. Then the average of $\log |f(z)|$ over the circle of radius R is given by:

$$\log |f(0)| + \sum_{f(z)=0;\ |z|<R} \log \frac{R}{|z|}.$$

    Proof: Suffices to assume $R = 1$. Clear if f has no zeros, because $\log |f(z)|$ is harmonic. Clear for a Blaschke factor $(z-a)/(1-\overline{a}z)$. But the formula is true for fg if it is true for f and g, so we are done.

    Cor: Let $n(r)$ be the number of zeros of f inside the circle of radius r. Then

$$\int_0^R n(r) \frac{dr}{r} = \frac{1}{2\pi} \int_0^{2\pi} \log |f(Re^{i\theta})| d\theta - \log |f(0)|.$$

    Remark: We have used the mean value property of harmonic functions. This holds for any harmonic function u on the disk by writing $u = \mathrm{Re}(f)$, f holomorphic, and then applying Cauchy's integral formula for $f(0)$.

    The physical idea of Jensen's formula is that $\log |f|$ is the potential for a set of unit point charges at the zeros of f.

58. Entire functions of finite order. An entire function $f : \mathbb{C} \to \mathbb{C}$ is of finite order if there is an $A > 0$ such that $|f(z)| = O(\exp |z|^A)$. The least such A is the *order* ρ of f.

    Examples: Polynomials have order 0; $\sin(z), \cos(z), \exp(z)$ have order 1; $\cos(\sqrt{z})$ has order 1/2; $\exp(\exp(z))$ has infinite order.

59. Number of zeros. By Jensen's formula, if f has order ρ, then $n(r) = O(r^{\rho+\epsilon})$, where $n(r)$ is the zero counting function for f. Corollary: $\sum 1/|a_i|^{\rho+\epsilon} < \infty$, where a_i enumerates the zeros of f (other than zero itself).

    In other words, $\rho(a_i) \le \rho(f)$, where $\rho(a_i)$ is the exponent of convergence of the zeros of f, i.e. the least ρ such that $\sum 1/|a_i|^{\rho+\epsilon} < \infty$.

60. Definition: a *canonical product* is an entire function of the form

$$f(z) = z^m \prod E_p(z/a_i)$$

    where p is the least integer such that $\sum |z/a_i|^{p+1} < \infty$ for all z.

61. Hadamard's Factorization Theorem. Let f be an entire function of order ρ. Then $f(z) = P(z) \exp Q(z)$, where P is a canonical product with the same zeros as f and Q is a polynomial of degree less than or equal to ρ.

62. To prove Hadamard's theorem we develop two estimates. First, we show a canonical product $P(z)$ is an entire function of order $\rho = \rho(a_i)$. This is the least order possible for the given zeros, by Jensen's theorem. Second, we show a canonical product has $m(r) \ge \exp(-|z|^{\rho+\epsilon})$ for most radii r, where $m(r)$ is the minimum of $|f(z)|$ on the circle of radius r. More precisely, for any $\epsilon > 0$, this lower bound holds for all r outside a set of finite total length. In particular it holds for r arbitrarily large.

    Given these facts, we observe that $f(z)/P(z)$ is an entire function of order ρ with *no zeros.* Thus $Q(z) = \log f(z)$ is an entire function satisfying $\operatorname{Re} Q(z) = O(1 + |z|^{\rho+\epsilon})$. This implies Q is a polynomial. Indeed, for any entire function $f(z) = \sum a_n z^n$, we have $|a_n| r^n \le \max(4A(r), 0) - 2\operatorname{Re}(f(0))$, where $A(r) = \max \operatorname{Re} f(z)$ over the circle of radius r.

63. Example:

$$\sin(\pi z) = \pi z \prod_{n \neq 0} \left(1 - \frac{z^2}{n^2}\right).$$

    Indeed, the right hand side is a canonical product, and $\sin(\pi z)$ has order one, so the formula is correct up to a factor $\exp Q(z)$ where $Q(z)$ has degree one. But since $\sin(\pi z)$ is odd, we conclude Q has degree zero, and by checking the derivative at $z = 0$ of both sides we get $Q = 0$.

64. Little Picard Theorem. A nonconstant entire function omits at most one value in the complex plane. (This is sharp as shown by the example of $\exp(z)$.)

    Great Picard Theorem. Near an essential singularity, a meromorphic function assumes all values on $\widehat{\mathbb{C}}$ with at most two exceptions.

    Proof of Little Picard: The key fact is that the universal cover of $\mathbb{C} - \{0, 1\}$ can be identified with the upper halfplane. This can be seen by constructing a Riemann mapping from the upper halfplane to an ideal hyperbolic triangle, sending 0, 1 and ∞ to the vertices, and then developing both the domain and range by Schwarz reflection. Indeed, with suitable normalizations we have that $\mathbb{C} - \{0, 1\}$ is isomorphic to $\mathbb{H}/\Gamma(2)$, where $\Gamma(2) \subset PSL_2(\mathbb{Z})$ is the group of matrices congruent to the identity modulo 2. This is a free group on two generators.

    Given this fact, we lift an entire function $f : \mathbb{C} \to \mathbb{C} - \{0, 1\}$ to a map $\widetilde{f} : \mathbb{C} \to \mathbb{H}$, which is constant by Liouville's theorem (because $\mathbb{H} \cong \Delta$).

65. Proof of Great Picard. Let $f : \Delta^* \to \mathbb{C} - 0, 1$ be a holomorphic function omitting three values on the sphere, normalized to be zero, one and infinity. Consider a loop γ around the puncture of the disk. If f sends γ to a contractible loop on the triply-punctured sphere, then f lifts to a map into the universal cover $\mathbb{H}$, which implies by Riemann's removability theorem that f extends holomorphically over the origin.

    Otherwise, by the Schwarz lemma, $f(\gamma)$ is a homotopy class that can be represented by an arbitrarily short loop. Thus it corresponds to a puncture, which we can normalize to be $z = 0$ (rather than 1 or ∞). It follows that f is bounded near $z = 0$ so again the singularity is not essential.

66. The Gamma Function:

$$\Gamma(z) = \frac{\exp(-\gamma z)}{z} \prod_1^\infty \left(1 + \frac{z}{n}\right)^{-1} \exp(z/n).$$

    Note that this expression contains the reciprocal of the canonical product associated to the non-positive integers. The constant γ is chosen so that $\Gamma(1) = 1$; it can be given by:

$$\gamma = \lim(\sum_1^n 1/k) - \log(n+1).$$

This expression is the error in an approximation to $\int_1^{n+1} dx/x$ by the area of n rectangles of base one lying over the graph.

67. Gauss's formula:

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}.$$

The functional equation:

$$\Gamma(z+1) = z\Gamma(z).$$

Corollary: $\Gamma(n+1) = n!$.

68. The integral representation: for $\mathrm{Re}(z) > 0$,

$$\Gamma(z) = \int_0^\infty e^{-t} t^z \frac{dt}{t}.$$

In other words, $\Gamma(z)$ is the *Mellin transform* of the function e^{-t} on $\mathbb{R}^*$. The Mellin transform is an integral against characters $\chi : \mathbb{R}^* \to \mathbb{C}^*$ (given by $\chi(t) = t^z$), and as such it can be compared to the Fourier transform (for the group $\mathbb{R}$ under addition) and to Gauss sums. Indeed the Gauss sum

$$\sigma(\chi) = \sum_{(n,p)=1} \chi(n) e^{2\pi i n/p}$$

is the analogue of the Gamma function for the group $(\mathbb{Z}/p)^*$.

69. Relation to sine function:

$$\frac{\pi}{\sin(\pi z)} = \Gamma(z)\Gamma(1-z).$$

Proof: Form the product $\Gamma(z)\Gamma(-z)$ and use the functional equation.

70. Periodic functions: If $f : \mathbb{C} \to X$ has period λ, then $f(z) = F(\exp(2\pi i z/\lambda)$, where $F : \mathbb{C}^* \to X$. Example:

$$\sum_{-\infty}^{\infty} \frac{1}{(z-n)^2} = \frac{\pi^2}{\sin^2(\pi z)}.$$

From this we get $\sum_1^\infty 1/n^2 = \pi^2/6$.

71. Elliptic (doubly-periodic) functions. Such functions can be considered as holomorphic maps $f : X \to \widehat{\mathbb{C}}$, where $X = \mathbb{C}/\Lambda$. By general easy results on compact Riemann surfaces: An entire elliptic function is constant. The sum of the residues of f over X is zero. The map f has as many zeros as poles.

More interesting is the fact that, if f is nonconstant with zeros a_i and poles p_i, then $\sum a_i = \sum p_i$ in the group law on X. (Proof: integrate $z df/f$ around a fundamental parallelogram in $\mathbb{C}$.)

We will see that we may *construct* an elliptic function with given zeros and poles subject only to this constraint.

72. Construction of elliptic functions. For $n \geq 3$,

$$\zeta_n(z) = \sum_{\Lambda} \frac{1}{(z-\lambda)^n}$$

defines an elliptic function of degree n. For degree two, we adjust the factors in the sum so they vanish at the origin, and obtain the definition of the Weierstrass function:

$$\wp(z) = \frac{1}{z^2} + \sum_{\Lambda}{}' \frac{1}{(z-\lambda)^2} - \frac{1}{\lambda}.$$

To see $\wp(z)$ is elliptic, use the fact that $\wp(z) = \wp(-z)$ and $\wp'(z) = -2\zeta_3(z)$ (which implies $\wp(z+\lambda) = \wp(z) + A_\lambda$).

Other ways to construct elliptic functions of degree two: if Λ is generated by (λ_1, λ_2), first sum over one period to get:

$$f_1(z) = \sum_{-\infty}^{\infty} \frac{1}{(z-n\lambda_1)^2} = \frac{\pi^2}{\lambda_1^2 \sin(\pi z/\lambda_1)^2};$$

then we have

$$\wp(z) = \sum_{-\infty}^{\infty} f_1(z - n\lambda_2)$$

(and the convergence is rapid). Similarly, if we write $X = \mathbb{C}^*/ < z \mapsto \alpha z >$, then

$$f(z) = \sum \frac{\alpha^n z}{(\alpha^n z - 1)^2}$$

converges to an elliptic function of order two.

73. Laurent expansion and differential equation. Expanding $1/(z-\lambda)^2$ in a Laurent series about $z = 0$ and summing, we obtain

$$\wp(z) = \frac{1}{z^2} + \sum_{1}^{\infty}(2n+1)z^{2n}G_n$$

where

$$G_n = \sum_{\Lambda}{}' \frac{1}{\lambda^{2n}}.$$

By a straightforward calculation (using the fact that an elliptic function with no pole is constant) we have:

$$\wp'(z)^2 = 4\wp(z)^3 - g_2\wp(z) - g_3,$$

where $g_2 = 60G_2$ and $g_3 = 140G_3$.

74. Geometry of the Weierstrass map $\wp : X \to \widehat{\mathbb{C}}$. The critical points of $\wp$ are the points of order two; the critical values are infinity and the zeros of the cubic equation $p(x) = 4x^3 - g_2x - g_3$. These are distinct.

The map $z \mapsto (\wp(z), \wp'(z))$ sends $X - \{0\}$ to the affine cubic curve $y^2 = p(x)$. Thus $\wp$ "uniformizes" this plane curve of genus 1.

When Λ has a rectangular fundamental domain, the $\wp$ function can also be constructed by taking a Riemann mapping from this rectangle (with dimensions reduced by a factor of two) to the upper half-plane and developing by Schwarz reflection. Then the differential equation for $\wp$ comes from the Schwarz-Christoffel formula.

75. Function fields. Given any Riemann surface X, the meromorphic functions on X form a field $K(X)$. Then K is a contravariant functor from category of Riemann surfaces with non-constant maps to the category of fields with extensions. Example: $K(\widehat{\mathbb{C}}) = \mathbb{C}(z)$.

Theorem: For $X = \mathbb{C}/\Lambda$, $K(X) = \mathbb{C}(x,y)/(y^2 - 4x^3 + g_2 x + g_3)$.

To see that $\wp$ and $\wp'$ generate $K(X)$ is easy. Any even function $f : X \to \widehat{\mathbb{C}}$ factors through $\wp$: $f(z) = F(\wp(z))$, and so lies in $\mathbb{C}(\wp)$. Any odd function becomes even when multiplied by $\wp'$; and any function is a sum of one even and one odd.

To see that the field is exactly that given is also easy. It amounts to showing that $K(X)$ is of degree exactly two over $\mathbb{C}(\wp)$, and $\wp$ is transcendental over $\mathbb{C}$. The first assertion is obvious, and if the second fails we would have $K(X) = \mathbb{C}(\wp)$, which is impossible because $\wp$ is even and $\wp'$ is odd.

76. An elliptic function with given poles and zeros. It is natural to try to construct an elliptic function by forming the Weierstrass product for the lattice Λ:

$$\sigma(z) = z \prod_{\Lambda}{}' \left(1 - \frac{z}{\lambda}\right) \exp\left(\frac{z}{\lambda} + \frac{z^2}{2\lambda^2}\right).$$

Then $(\sigma'(z)/\sigma(z))' = -\wp(z)$, from which it follows that $\sigma(z+\lambda) = \sigma(z)\exp(a_\lambda + b_\lambda z)$. From this it is easy to see that

$$\frac{\sigma(z-a_1)\dots\sigma(z-a_n)}{\sigma(z-p_1)\dots\sigma(z-p_n)}$$

defines an elliptic function whenever $\sum a_i = \sum p_i$, and therefore this is the only condition imposed on the zeros and poles of an elliptic function.

77. The *moduli space of elliptic curves*, $\mathcal{M}_1$, classifies Riemann surfaces of genus 1 up to isomorphism. Since any such surface is given by $X = \mathbb{C}/\Lambda$, and we may normalize so $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau$, where $\tau \in \mathbb{H}$, it is not hard to see:

$$\mathcal{M}_1 = \mathbb{H}/PSL_2(\mathbb{Z}).$$

Similarly, an elliptic curve with a labelling of its points of order two is classified by $\mathcal{M}_1(2) = \mathbb{H}/\Gamma(2)$.

We now present another proof that $\mathbb{H}/\Gamma(2)$ is naturally isomorphic to $\widehat{\mathbb{C}} - \{0, 1, \infty\}$. Given Λ, we get the cubic polynomial $4x^3 - g_2 x - g_3 = 4(x - x_1)(x - x_2)(x - x_3)$, whose roots are exactly the values of $\wp(z)$ at the points of order two. Thus the cross-ratio $\lambda(\tau) = (x_3 - x_2)/(x_3 - x_1)$ depends only on the location of τ in $\mathcal{M}_1(2)$, and thus we have a map $\lambda : \mathcal{M}_1(2) \to \widehat{\mathbb{C}} - \{0, 1, \infty\}$.

But we can invert this map: given λ, form the degree two cover X_λ of $\widehat{\mathbb{C}}$ branched over $\{0, 1, \infty, \lambda\}$. Once we know X_λ is isomorphic to $\mathbb{C}/\Lambda$ for some Λ, it is easy to show we have reconstructed τ up to an element of $\Gamma(2)$. One might appeal to the uniformization theorem here, but in fact it is elementary to see that $dx/\sqrt{x(x-1)(x-\lambda)}$ lifts to a nowhere-zero holomorphic 1-form on X_λ. This one form gives a complete Euclidean metric, and so the universal cover of X_λ can be identified with $\mathbb{C}$.