Dynamics on Riemann surfaces and the geometry of moduli space

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\[ \frac{\mathbb{H}/\Gamma}{\mathbb{H}} \rightarrow \mathcal{M}_g \rightarrow \mathbb{B}/\!\!\!/\Gamma \]

Avila, Hubert, Lanneau, Kontsevich, Masur, Yoccoz, Zorich, ...

Hyperbolic surfaces

\[ X \text{ genus } g \]
Riemann surface

Hyperbolic metric \( \rho^2(z) |dz|^2 \)
- Geodesic flow on \( T_1 X \)
- Ergodic, mixing, entropy > 0
- \# Loops \( L(C) < L \sim e^{L}/L \)
- \( \text{Aut}(X, \rho) \) finite
- Charts into \( \mathbb{H} \)

Flat metrics

\((X, \omega)\)

\[ \Omega(X) = \{ \text{holomorphic forms } \omega(z) \, dz \} = \mathbb{C}^g \]

\( \omega \in \Omega(X) \Rightarrow \) flat metric \(|\omega|^2 |dz|^2 \)

\( \omega(p) = 0 : \) negatively curved singularities

- Geodesics with fixed slope foliate \( X \)
- Not mixing, entropy = 0
- \# Smooth Loops \( L(C) < L \sim L^2 \)
- Charts into \( \mathbb{C} = \mathbb{R}^2 \) up to translation

Example: Billiards

\( (X, \omega) = \left( \bigcup \rho P, dz \right) / \sim \)

\( X \text{ has genus } 2 \)
\( \omega \text{ has just one zero!} \)
**Real Symmetries**

\[ \text{Aff}(X, \omega) = \{ f : X \to X, \text{real linear maps} \} \]

\[ \text{SL}(X, \omega) = \{ Df \} = \Gamma \subset \text{SL}_2(\mathbb{R}) \quad \text{discrete} \]

**Theorem (Veech):** If \( \text{SL}(X, \omega) \) is a lattice, then the geodesic flow has optimal dynamics.

**Proof:** renormalization.

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**Optimal Billiards**

**Example:** if \( X = \mathbb{C}/\Lambda, \ \omega = dz \), then \( \text{SL}(X, \omega) = \text{SL}_2(\mathbb{Z}) \)

**Theorem (Veech, 1989):** For \((X, \omega) = (y^2 = x^n - 1, dx/ly)\), \( \text{SL}(X, \omega) \) is a lattice.

**Corollary**

A billiard path in a regular polygon is periodic or uniformly distributed.

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**Moduli space perspective**

\( \mathcal{M}_g \) = the moduli space of Riemann surfaces \( X \) of genus \( g \)

\[ \left\{ \begin{array}{c}
\quad \\
\end{array} \right\} \quad \text{-- a complex variety, dimension } 3g-3 \]

**Teichmüller metric:** every holomorphic map \( f : \mathbb{H}^2 \to \mathcal{M}_g \) is distance-decreasing.

= *Kobayashi metric*

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**Dynamics over moduli space**

\( \Omega \mathcal{M}_g \) = space of holomorphic 1-forms \( (X, \omega) \) of genus \( g \)

\( \mathcal{M}_g \)

\( \text{SL}_2(\mathbb{R}) \) acts on \( \Omega \mathcal{M}_g \)

Stabilizer of \((X, \omega) = \text{SL}(X, \omega)\)
Teichmüller curves

\[ \text{SL}_2(\mathbb{R}) \text{ orbit of } (X, \omega) \text{ in } \Omega \mathcal{M}_g \text{ projects to a complex geodesic in } \mathcal{M}_g: \]

\[ \mathbb{H} \rightarrow \mathcal{M}_g \]
\[ \downarrow \quad f \]
\[ V = \mathbb{H} / \text{SL}(X, \omega) \]

\[ \text{SL}(X, \omega) \text{ lattice } \Leftrightarrow f : V \rightarrow \mathcal{M}_g \text{ is an algebraic, isometrically immersed Teichmüller curve.} \]

20th century lattice billiards

<table>
<thead>
<tr>
<th>Shape</th>
<th>Group</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td>SL₂(\mathbb{Z})</td>
</tr>
<tr>
<td>Tiled by squares</td>
<td>~ SL₂(\mathbb{Z})</td>
</tr>
<tr>
<td>Regular polygons</td>
<td>~ (2, n, \infty) triangle group</td>
</tr>
<tr>
<td>Various triangles</td>
<td>triangle groups</td>
</tr>
</tbody>
</table>

Explicit package: Pentagon example

\( V = \mathbb{H}/\text{SL}(X, \omega) \subset \text{SL}_2(\sqrt{5}) \)

\[ = \langle a, b \rangle \]

⇒ Direct proof that SL(X, \omega) is a lattice

Genus 2

\[ \leadsto \text{Regular 5-8- and 10-gon} \]

Problem

Are there infinitely many primitive Teichmüller curves V in the moduli space \( \mathcal{M}_2 \)?
Jacobiens with real multiplication

**Theorem**

\((X, \omega)\) generates a Teichmüller curve \(V\) if \(\text{Jac}(X)\) admits real multiplication by \(\mathcal{O}_D \subset \mathbb{Q}(\sqrt{D})\).

**Corollary**

\(V\) lies on a Hilbert modular surface

\[ V \subset \mathcal{H}_D \subset \mathcal{M}_2 \]

\(\mathcal{H} \times \mathcal{H} / \text{SL}_2(\mathcal{O}_D)\)

**Idea of Proof:** \(f + f^{-1}\) acts on \(\text{Jac}(X)\) \(\Rightarrow\)

trace ring of \(\text{SL}(X, \omega)\) acts

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The Weierstrass curves

\[ W_D = \{ X \in \mathcal{M}_2 : \text{Jac}(X) \text{ admits real multiplication by } \mathcal{O}_D \text{ with an eigenform } \omega \text{ with a double zero.} \} \]

**Theorem.** \(W_D\) is a finite union of Teichmüller curves.

\[ \begin{array}{c}
\text{P}_d \\
\gamma = (1 + \sqrt{d})/2 \\
\end{array} \]

**Corollaries**

- \(P_d\) has optimal billiards for all integers \(d > 0\).
- There are infinitely many primitive \(V\) in genus 2.

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Examples of Weierstrass curves

\(W_D \supset \mathbb{H}/\Gamma\)

\(\Gamma\) not arithmetic

no known direct description of \(\Gamma\)

\(\leftarrow\) algorithm only works if \(g(W_D) = 0\)

- **Components of** \(W_D\)

**Theorem.** \(W_D\) is irreducible unless \(D = 1 \mod 8\) and \(D > 9\), in which case it has two components.

**Proof**

a) combinatorial enumeration of cusps of \(W_D\)

b) elementary move relating cusps in same component \(\Rightarrow\) graph \(S_D\)

c) proof that \(S_D\) is connected

(translate: analytic number theory + computer for \(D < 2000\))
Euler characteristic of $W_D$

Theorem (Bainbridge, 2006)

$$\chi(W_D) = -\frac{9}{2} \chi(SL_2(\mathcal{O}_D))$$

= coefficients of a modular form

Compare: $\chi(M_{g,1}) = \zeta(1-2g)$ (Harer-Zagier)

Proof: Uses cusp form on Hilbert modular surface with $(\alpha) = W_D - P_D$, where $P_D$ is a Shimura curve

Elliptic points on $W_D$

Theorem (Mukamel, 2011)

The number of orbifold points (of order 2) on $W_D$ is given by a sum of class numbers for $\mathbb{Q}(\sqrt{-D})$.

Proof: $(X, \omega)$ corresponds to an orbifold point ⇒ $X$ covers a CM elliptic curve $E$ ⇒ $(X, \omega), p: X \to E$ and $\text{Jac}(X)$ can be described explicitly.

Explicit points on $W_D$

$$X \in M_2$$

D=5 $y^2 = x^5 - 1$

D=8 $y^2 = x^8 - 1$

D=13 $y^2 = (x^2 - 1)(x^4 - ax^2 + 1)$

$a = 2594 + 720 \sqrt{13}$

Mukamel

D=108 $96001 + 48003 a + 3 a^2 + a^3 = 0$

Genus of $W_D$

<table>
<thead>
<tr>
<th>D</th>
<th>$g(W_D)$</th>
<th>$c_2(W_D)$</th>
<th>$C(W_D)$</th>
<th>$\chi(W_D)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0 1 1</td>
<td>-$\frac{1}{2}$</td>
<td>15</td>
<td>-15</td>
</tr>
<tr>
<td>8</td>
<td>0 0 2</td>
<td>-$\frac{1}{4}$</td>
<td>7</td>
<td>-$\frac{1}{4}$</td>
</tr>
<tr>
<td>9</td>
<td>0 1 2</td>
<td>-$\frac{1}{4}$</td>
<td>10</td>
<td>-15</td>
</tr>
<tr>
<td>12</td>
<td>0 1 3</td>
<td>-$\frac{1}{4}$</td>
<td>12</td>
<td>-18</td>
</tr>
<tr>
<td>13</td>
<td>0 1 3</td>
<td>-$\frac{1}{4}$</td>
<td>12</td>
<td>-18</td>
</tr>
<tr>
<td>16</td>
<td>0 1 3</td>
<td>-$\frac{1}{4}$</td>
<td>13</td>
<td>-$\frac{1}{4}$</td>
</tr>
<tr>
<td>17</td>
<td>(0,0) (1,1) (3,3)</td>
<td>($-\frac{1}{2}$, $-\frac{1}{2}$)</td>
<td>17</td>
<td>-18</td>
</tr>
<tr>
<td>20</td>
<td>0 0 5</td>
<td>-3</td>
<td></td>
<td></td>
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<tr>
<td>21</td>
<td>0 2 4</td>
<td>-3</td>
<td></td>
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<tr>
<td>24</td>
<td>0 1 6</td>
<td>-2</td>
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<tr>
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<td>(0,0) (0,1) (5,3)</td>
<td>($-\frac{1}{2}$, $-\frac{1}{2}$)</td>
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<tr>
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<td>0 2 7</td>
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<td>33</td>
<td>(0,0) (1,1) (6,6)</td>
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<td>(0,0) (2,2) (7,7)</td>
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<tr>
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<tr>
<td>49</td>
<td>(0,0) (2,0) (10,8)</td>
<td>($-\frac{1}{2}$, $-\frac{1}{2}$)</td>
<td>49</td>
<td>-$\frac{1}{2}$</td>
</tr>
</tbody>
</table>

Corollary

$W_D$ has genus 0 only for $D < 50$

( table by Mukamel)
**Genus 3 or more**

*Algebraically primitive case:*
Trace field $K$ of $\text{SL}(X, \omega)$ has degree $g=g(X)$ over $\mathbb{Q}$.
(avoid echoes of lower genera)

*Theorem (Möller)*
- $\text{Jac}(X)$ admits real multiplication by $K$,
- $P-Q$ is torsion in $\text{Jac}(X)$ for any two zeros of $\omega$.

*Methods:* Variation of Hodge structure; rigidity theorems of Deligne and Schmid; Neron models; arithmetic geometry

**Finitness conjecture:**
There are only finitely many algebraically primitive Teichmüller curves in $\mathcal{M}_g$, $g=3$ or more.

*Theorem (Möller, Bainbridge-Möller)*
Holds for hyperelliptic stratum $(g-1,g-1)$
Holds for $g=3$ stratum $(3,1)$

**Conjectures on dynamics on $\Omega \mathcal{M}_g$**

I. Every $\text{SL}_2(\mathbb{R})$ orbit-closure and every $\text{SL}_2(\mathbb{R})$ ergodic measure on $\Omega \mathcal{M}_g$ is algebraic.

II. The closure of any complex geodesic $f(\mathbb{H})$ is an algebraic subvariety of $\mathcal{M}_g$.

*Celebrated theorem of Ratner (1995) →*
true for $\text{SL}_2(\mathbb{R})$ acting on $G/\Gamma$

**Genus two**

*Theorem* These conjectures hold for genus $g=2$.

*Proof:* 1) Any 1-form $(X, \omega)$ of genus 2 is a connect sum of forms of genus 1

2) Ratner’s theorem holds for diagonal unipotent actions on $\text{SL}_2(\mathbb{R}) \times \text{SL}_2(\mathbb{R}) / \text{SL}_2(\mathbb{Z}) \times \text{SL}_2(\mathbb{Z})$. 
Complex geodesics in genus two

Theorem

Let \( f : \mathbb{H} \to \mathcal{M}_2 \) be a complex geodesic generated by \( (X, \omega) \in \Omega \mathcal{M}_g \). Then \( f(\mathbb{H}) \) is either:

- A Teichmüller curve (such as \( \mathcal{W}_D \)), \( \dim = 1 \)
- A Hilbert modular surface \( \mathcal{H}_D \), or \( \dim = 2 \)
- The whole space \( \mathcal{M}_2 \), \( \dim = 3 \)

Hilbert modular surface in \( \mathcal{M}_2 \) is foliated by complex geodesics \( = \mathbb{H} \)

How \( \mathcal{W}_5 \) sits on \( \mathcal{H}_5 \)

The universal cover of \( \mathcal{W}_5 \) = the graph of \( F \) in the universal cover \( \mathbb{H} \times \mathbb{H} \) of \( \mathcal{H}_5 \)

(closed leaf of the foliation)

Dynamics on \( \Omega \mathcal{M}_g, \ g \geq 2 \)

Conjecture

1. Every \( \text{SL}_2(\mathbb{R}) \) orbit-closure and every \( \text{SL}_2(\mathbb{R}) \) ergodic measure on \( \Omega \mathcal{M}_g \) is algebraic.

Measure case: recent progress by Eskin and Mirzakhani: Clay Meeting 16 May 2011

Orbit closures still open(?)
**Complex geometry of moduli space**

B \subset \mathbb{C}^n \text{ bounded domain}

*Kobayashi metric*: Every holomorphic map \( \mathbb{H} \to (B, g_K) \) is contracting

*Carathéodory metric*: Every holomorphic map \( (B, g_C) \to \mathbb{H} \) is contracting

**Theorem**: \( B = G/K \) a symmetric domain \( \Rightarrow g_K = g_C \)

**Proof**: B has a convex model

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**Bers embedding**

\[ \mathcal{M}_g = \mathcal{T}_g / \text{Mod}_g \]

\( \mathcal{T}_g \hookrightarrow \mathbb{C}^{3g-3} \) as a bounded domain

**Open Problem**: Does \( g_K = g_C \) on Teichmüller space?

**Image of** \( \mathcal{T}_g \)

**Lots of cusps**

*(Dumas)*

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**Related Questions**

1. Does \( \mathcal{M}_g \) embed isometrically into an infinite product of locally symmetric spaces?

   *(for the Kobayashi metric)*

2. Is the super period map \( \mathcal{M}_g \to \prod \gamma_{\mathcal{A}_h} \) an isometric embedding?

   \( X \to (\text{Jac}(Y) : Y \text{ is a finite cover of } X) \)

**Theorem (Kra)**

The super period map is an isometry on all complex geodesics \( \mathbb{H} \to \mathcal{M}_g \) generated by 1-forms \( (X, \omega) \).

*[Hence on all \( T \) curves \( V \subset \mathcal{M}_g \)]

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**Kazhdan’s Theorem**

\[ \mathbb{H} \]

\[ \downarrow \]

\[ Y \to \text{Jac}(Y) = \mathbb{C}^h / \Lambda \]

\[ \downarrow \]

\[ X \]

The hyperbolic metric on \( X \) is the limit of the metrics inherited from the Jacobians of finite covers of \( X \).
However...

Theorem.
The super period map
\( M_g \rightarrow \prod C_h \)
is not an isometry in the directions coming from quadratic differentials with odd order zeros.

Corollary.
The entropy of most mapping classes \( f : \Sigma_g \rightarrow \Sigma_g \) cannot be detected homologically, even after passing to finite covers.

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entropy on topological surfaces

\[ \text{Mod}_g = \{ f : \Sigma_g \rightarrow \Sigma_g \}/\text{isotopy} = \pi_1(\mathcal{M}_g) \]

\[ h(f) = \min \{ \text{entropy of } g : g \text{ isotopic to } f \} \]

\[ = \text{length of loop on moduli space represented by } [f] \]

\[ \geq \log \text{spectral radius of } f^* \text{ on } H^1(\Sigma_g) \]

\( \Sigma_h \rightarrow \Sigma_h \)

\( \downarrow \quad \downarrow \)

\( \Sigma_g \rightarrow \Sigma_g \)

\[ \sup \log \text{spectral radius of } F^* \text{ on } H^1(\Sigma_h), \]

over all finite covers.

topological proof? Koberda