Solving polynomials: braids, rigidity and dynamics

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Complex dynamics

Bifurcations

Lattès examples
Rigidity

An algebraic family of rational maps is either trivial, Lattès, or has bifurcations.

How to compute the cube root of 2

The problem is to determine the regions of the plane such that $P$, taken at pleasure anywhere within one region, we arrive ultimately at the point $A$, anywhere within another region we arrive at the point $B$, and so for the several points representing the root of the equation. The solution is easy and elegant for the case of a quadric equation; but the next succeeding case of a cubic equation appears to present considerable difficulty.
The space of cubics
Unsolvability of the quartic

There is no purely iterative algorithm to solve polynomials of degree 4 or more.

What about cubics?

$S_3$ symmetry

$f(z) = (z^4 + 2z)/(2z^3 + 1)$

How to solve cubics

II. There exists a unique degree 4 superconvergent algorithm for cubics. If the cubic polynomial $p$ is given by

\[ p(X) = X^3 + aX + b \]

then the algorithm is given by

\[ T_p(X) = X - \frac{(X^3 + aX + b)(3aX^2 + 9bX - a^2)}{(3aX^3 + 18bX^2 - 6a^2X^2 - 6abX - 9b^2 - a^3)}. \]

= (Tate) Newton for

\[ q(X) = \frac{p(X)}{(3aX^2 + 9bX - a^2)}. \]
Braids

The braiding of the attractor in a stable family of rational maps is either

- reducible,
- finite, or
- it fixes a point of the attractor.

Location of failures

Every algorithm fails somewhere along this loop in the space of degree 4 polynomials

Fig. 2.1. This braid does not arise for rational maps

Solving polynomials through the ages

Various authors

Solving the quadratic, circa 2000 BC

Solving the cubic, circa 1500 AD

Solving the quartic, circa 1500 AD

Insolvability of the quintic 1824

Towers of algorithms

A field extension can be computed by a tower of purely iterative algorithms iff its Galois group is within $A_5$ of solvable.
Solvability of the quintic

The quintic can be solved by a tower of algorithms, but the sextic cannot.

Rational maps with $A_5$ symmetry

![Diagram of geometric construction of a rational map]

Vertices, face centers, edge midpoints

$$f = x^{11} + 11x^6y^5 - xy^{11}$$
$$H = -x^{20} - y^{20} + 228(x^{15}y^5 - x^5y^{15}) - 494x^{10}y^{10}$$
$$T = x^{30} + y^{30} + 522(x^{25}y^5 - x^5y^{25}) - 10005(x^{20}y^{10} + y^{10}x^{20})$$

**Proposition 5.3.** There are exactly four rational maps of degree $<31$ which commute with the icosahedral group. These four maps, of degree 1, 11, 19 and 29 respectively, are:

- $f_1(z) = z$
- $f_2(z) = \frac{z^{14} + 66z^6 - 11z}{-11z^{10} - 66z^5 + 1}$
- $f_3(z) = \frac{-57z^{15} + 247z^{10} + 171z^5 + 1}{-z^{19} + 171z^{14} - 247z^9 - 57z^4}$
- $f_4(z) = \frac{-87z^{25} - 335z^{20} - 6670z^{10} - 435z^5 + 1}{-z^{29} - 435z^{24} + 6670z^{19} + 3335z^9 + 87z^4}$

Fig. 4. Geometric construction of a rational map.
Degree 6

Can polynomials of degree 6 be solved using algebraic functions of just one variable?

Hilbert’s 13th problem