Rigidity for planes in hyperbolic 3-manifolds

**Rigidity of planes**

Hyperbolic plane $\mathbb{H}^2$ \( \mathbb{M} = \mathbb{H}^3/\Gamma \)

image = immersed plane $\mathbb{P} \subset \mathbb{M}$

*Shah, Ratner*

$\mathbb{M}$ is compact $\Rightarrow \mathbb{P}$ is dense (typical) or

$\mathbb{P}$ is a closed surface (miracle)

**Totally geodesic planes**

- Euclidean Planes $\mathbb{R}^2$ (space of lattices $L$ in $\mathbb{R}^3$) \( \text{Littlewood} \)
- Hyperbolic Planes $\mathbb{H}^2$ (space of lattices $L$ in $\mathbb{R}^3$) \( \text{Oppenheim} \)
- Hyperbolic Planes $\mathbb{H}^2$ (moduli space $\mathbb{M}_g$) \( \text{Teichmüller} \)
- Hyperbolic Planes $\mathbb{H}^2$ (3-manifold $\mathbb{H}^3/\Gamma$) \( \text{Ratner-Shah / Thurston} \)

**Infinite volume 3-manifolds**

*Does rigidity persist??*

No. For $\mathbb{M} = S \times \mathbb{R}$, cylinders cause problems

*M, Mohammadi, Oh*

Yes. Rigidity persists for acylindrical 3-manifolds.

*Topology of 3-manifolds enters*
Finite volume 2-manifolds
Horocycles are closed or dense \textit{Hedlund}

Infinite volume 2-manifolds
Horocycles are closed or dense \textit{Dalbo}

Infinite volume 2-manifolds
Horocycles are closed or dense \textit{Dalbo}

Finite volume 3-manifolds
$S^2 =$ boundary of $\mathbb{H}^3$
$C =$ boundary of plane
Finite volume 3-manifolds

\[ S^2 = \text{boundary of} \ H^3 \]
\[ C = \text{boundary of plane} \]
\[ M = \Gamma \setminus H^3 \]
\[ \Gamma \cdot C \]

Another immersed plane

\[ S^2 = \text{boundary of} \ H^3 \]
\[ C = \text{boundary of plane} \]
\[ M = \Gamma \setminus H^3 \]
\[ \Gamma \cdot C \]

Immersed plane

\[ S^2 = \text{boundary of} \ H^3 \]
\[ C = \text{boundary of plane} \]
\[ M = \Gamma \setminus H^3 \]
\[ \Gamma \cdot C \]

Dense plane in \( M \)

\[ S^2 = \text{boundary of} \ H^3 \]
Closed, totally geodesic surface in $M$

$S^2$ = boundary of $H^3$

Finite volume $M$

Arithmetic $M$: $SL_2(\mathbb{Z}[i])$

Compact plane in $M$
Open problem: Do infinitely many closed geodesic surfaces yield arithmetic \( \Rightarrow M \) is arithmetic?
Incised Torus Wild Sphere, of ;r, in black steatite. "What most people think of as a sphere, the surface of a ball-like object, is distinguished by some as a tame sphere. There are wild spheres. The difference is in how the sphere is embedded into three dimensional space; context is important. That is, the homotopy group of loops of the exterior of the embedded sphere, $\pi_1$, of one embedding need not be $\pi_1$ of another embedding. The idea of $A \neq A$ is very common in everyday language: think of your name, say $A$, now think of someone else named $A$, so $A \neq A$. Here the bifurcation stages yield two trunks, four arms, and then eight fingers. Note the long extension of stone of the arms."

Thurston's Knotted Wye Hyperbolic Space, in Carrara white marble. "Thurston describes the mathematical antecedent of this sculpture as the simplest hyperbolic three manifold with totally geodesic boundary. Pendent under the trefoil-like loops is the vertex of a wye the arms of which rise, link a neighbor, and descend below the wye foot-like; all three arms join the base to form a second wye. The evacuated spheroids stippling the surface of this marble recall the semicircle, hemisphere geodesic constructions of a hyperbolic three space. Because of the unusual extension achieved in this direct carving, the marble, when lightly tapped with a knuckle, rings like a bell."

EDITORIAL NOTE: The two photographs of Ferguson's sculptures above are transposed.

$3$-manifolds with geodesic boundary

$M = S^3 - Y$

Sierpinski carpets

$\Lambda = S^2 \setminus \bigcup D_i$

diam$(D_i) \to 0$

$1$ dim Cantor sets
If $M$ is convex cocompact and acylindrical, then any plane $P^*$ in $M^*$ is closed or dense.

$P^*$ is a properly immersed surface in $M^*$, $\pi_1(P^*)$ is nonelementary.
There are at most countably many such planes. If infinitely many, then $P_i^* \rightarrow M^*$ (equidistribution).
Proof: Slices of Carpets

Key: Thick carpets have thick slices

Coda: Surface groups

Limit set and fundamental domain

Delicately chosen plane
Q. Even in the case of compact $M$

what are the possibilities for the fundamental group of $P$?