Riemann Surfaces in
Dynamics, Topology and Arithmetic

II. The shape of moduli space

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Dynamical systems

60’s optimism. Conjecture: for any compact manifold $M$, the set of well-understood dynamical systems

$$f : M \to M$$

is open and dense in $\text{Diff}(M)$.

Inspiration: triumphs of transversality and differential topology. But false!

70’s realism. Is there at least one map $f$ with comprehensible dynamics in each component of $\text{Diff}(M)$?
Surfaces

$S =$ smooth, oriented closed surface of genus $g \geq 0$.

**Mapping class group:**

$$\text{Mod}(S) = \text{(diffeomorphisms of } S)/\text{isotopy}$$

$$= \text{(components of } \text{Diff}(S))$$

**Goal:**

*Classify* elements of $\text{Mod}(S)$

Find *understandable* representatives in each isotopy class

**Method:** Find spaces on which $\text{Mod}(S)$ acts.
Simple closed curves

\[ S = (\text{simple closed curves on } S)/\text{isotopy} \]

Intersection pairing:

\[ i : S \times S \to \{0, 1, 2, 3, 4, \ldots\}; \]

\[ i(\alpha, \beta) = \text{minimal number of intersections possible.} \]

**Example:**

\[ i(\alpha, \beta) = 2 \]

\[ \text{Mod}(S) \text{ acts on } S \text{ preserving } i(\cdot, \cdot) \]
The case of a torus (genus 1)

- $\text{Mod}(S)$ acts faithfully on $H_1(S, \mathbb{Z}) \cong \mathbb{Z}^2$
- $\text{Mod}(S) \cong SL_2(\mathbb{Z})$
- $S = \mathbb{P}H_1(S, \mathbb{Q}) \cong \mathbb{P}(\mathbb{Q}^2) \subset \mathbb{RP}^1 = \mathbb{R} \cup \{\infty\}$
- $\text{Mod}(S)$ acts on $S$ by Möbius transformations, preserving

\[
i \begin{pmatrix} p & r \\ q & s \end{pmatrix} = \left| \det \begin{pmatrix} p & r \\ q & s \end{pmatrix} \right|.
\]
Classification of maps on a torus

The three types of $f \in \text{Mod}(S)$, for $S$ a torus:

1. **Finite order.**

   $f_* : H_1(S, \mathbb{R}) \to H_1(S, \mathbb{R})$ has complex eigenvalues;
   $f^n = \text{id}$, some $n > 1$.

2. **Reducible.**

   $f_*$ has a multiple eigenvalue $(\pm 1)$;
   $f(\alpha) = \alpha$, some $\alpha \in S$;
   $f$ is a Dehn twist around $\alpha$.

3. **Anosov.**

   $f_*$ has real eigenvalues $K^{\pm 1}$;
   $i(f^n(\alpha), \beta) \asymp K^n \to \infty$;
   $f \sim F$, an area preserving, linear map;
   $F$ preserves a pair of irrational foliations of $S$. 
Anosov example

The curve $F^4(\alpha)$ for $\alpha = (1, 0)$ and $F = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. 
Higher genus

Theorem (Classification of Surface Diffeomorphisms)
Any \( f \in \text{Mod}(S) \) is finite order, reducible, or pseudo-Anosov.

—Thurston, 1979; Teichmüller, Nielsen, Bers

**Reducible:** there are disjoint simple curves \( \alpha_1, \ldots, \alpha_n \in S \) permuted by \( f \).

**Pseudo-Anosov:** for some \( K > 1 \), \( i(f^n(\alpha), \beta) \asymp K^n \).

(In fact \( f \sim F \) locally like \( F(x, y) = (Kx, K^{-1}y) \), preserving a pair of orthogonal foliations on \( S \).)
Hyperbolic Riemann surfaces

Universal cover.
Any compact Riemann surface $X$ of genus $g \geq 2$ is covered by the unit disk $\Delta$.

Hyperbolic metric. $X$ has a natural metric of curvature $-1$, coming from the invariant metric $2|dz|/(1 - |z|^2)$ on the disk.
Finiteness of automorphisms

**Theorem** For any compact Riemann surface $X$ of genus $g \geq 2$, the group of holomorphic maps $F : X \to X$ is finite.

**Proof.**

- $F$ is a hyperbolic isometry $\implies$ Aut($X$) is compact.
- If $F_1$ is close to $F_2$, then $F_1$ and $F_2$ are homotopic.
- Then $\tilde{F}_1 = \tilde{F}_2$ on $S^1$, since
  \[
  \frac{\text{(Euclidean metric)}}{\text{(Hyperbolic metric)}} \to 0 \quad \text{at } \partial \Delta = S^1.
  \]
  $\implies$ $F_1 = F_2$.
- Thus Aut($X$) is discrete.
- Discrete + compact $\implies$ finite.

\[\blacksquare\]

**Corollary** If $F : X \to X$ is conformal, then $F$ has finite order in Mod($F$).
Teichmüller space

$\text{Teich}(S) = \text{(space of Riemann surfaces } X \text{ marked by } S)$. 

A **marking** of $X$ by $S$ is a homotopy class of homeomorphism $h : S \rightarrow X$. 

**Theorem** $\text{Teich}(S)$ is homeomorphic to $\mathbb{R}^{6g-6}$. 

$\text{Mod}(S)$ acts on $\text{Teich}(S)$ by changing the marking: $f \circ S \overset{h}{\rightarrow} X$. 

**Moduli space:** 

$\mathcal{M}(S) = \mathcal{M}_g$

$= \text{Teich}(S)/\text{Mod}(S)$

$= (\text{space of all Riemann surfaces } X \cong S)$. 

$\pi_1(\mathcal{M}(S)) = \text{Mod}(S)$
Strategy

To analyze $f \in \text{Mod}(S)$,

*Search for a fixed-point in Teich($S$).*

\[ f \text{ has a fixed-point } X \in \text{Teich}(S) \implies \\
 f \sim F \in \text{Aut}(X) \implies \\
f \text{ has finite order.} \]
Short geodesics

Length function \( \ell : S \times \text{Teich}(S) \to \mathbb{R} \): \[
\ell_\alpha(X) = \text{(length of closed geodesic } \sim \alpha \text{ on } X).\]

**Theorem** There are at most \( 3g - 3 \) short geodesics on \( X \), and they are all simple and disjoint.

**Theorem (Mumford)** The set of Riemann surfaces \( X \) in moduli space \( \mathcal{M}(S) \) with \[
\inf_\alpha \ell_\alpha(X) \geq r > 0
\]
is compact.
Maps with minimal squeeze

Theorem (Teichmüller) For any $X, Y \in \text{Teich}(S)$, there exists a unique map

$$G : X \to Y,$$

respecting markings, with minimal conformal distortion.

Local form:

$$G(x + iy) = Kx + iK^{-1}y$$

in suitable complex coordinates ($K > 1$).

Teichmüller metric on $\text{Teich}(S)$:

$$d(X, Y) = \frac{1}{2} \log K.$$ 

$\text{Mod}(S)$ acts by isometries on $(\text{Teich}(S), d)$. 
Proof of the classification

Length of a mapping class:

$$\tau(f) = \inf \{ \text{length}(\gamma \subset \mathcal{M}(S)) : \gamma \sim [f] \in \pi_1 \mathcal{M}(S) \}.$$  

3 cases:

\[\text{not realized}\]
Geometric cases

Case 1. $\tau(f) = 0$, achieved. Then:

$$f \cdot X = X \text{ for some } X; \implies f \text{ is of finite order.}$$

Case 2. $\tau(f) > 0$, achieved. Then:

$F : X \to X$, Teichmüller map with $K(F)$ minimized, preserves its stretch foliations; \implies

$$i(f^n(\alpha), \beta) \asymp K^n$$

and $f \sim F$ is pseudo-Anosov.
Case of short geodesics

**Case 3.** \( \tau(f) \) not achieved. Then:

- We have loops \( \gamma_n \to \infty \) in moduli space \( \mathcal{M}(S) \);
- \( \text{length}(\gamma_n) \approx \tau(f), \gamma_n = [f] \in \pi_1 \).
- Mumford’s compactness theorem \( \Rightarrow \) for \( n \gg 0 \), \( X_n \in \gamma_n \) has disjoint short geodesics
  \[ \{\alpha_1, \ldots, \alpha_m\} \subset \mathcal{S}; \]
- \( \text{length}(\gamma_n) \) bounded \( \Rightarrow \) same geodesics short on \( f \cdot X_n \)
- \( \Rightarrow \) \( f \) is reducible.
Reprise: Moduli space for genus $g = 1$
Moduli space versus hyperbolic space

Let $G$ be a discrete subgroup of $\text{Isom}(\mathbb{H}^n)$.

**Classification of isometries.** Any $g \in G$ is either

1. **Elliptic:**
   $g$ has a fixed-point in $\mathbb{H}^n \implies g$ is of finite order; or

2. **Hyperbolic:**
   $g$ translates along a geodesic $\mathbb{H}^n$; or

3. **Parabolic:**
   $g$ has a unique fixed-point $p \in S_{\infty}^{n-1} = \partial \mathbb{H}^n$.

   - Compare finite order, pseudo-Anosov and reducible.
   - Just as $\partial \mathbb{H}^n = S_{\infty}^{n-1}$, we have
     $$\partial \text{Teich}(S) = \mathbb{PML}(S) \cong S^{6g-7}.$$
Failure of hyperbolicity

Theorem For $g > 1$, the moduli space $\mathcal{M}_g$ admits no metric of pinched negative curvature ($-a < K < -b < 0$).

Proof. The fundamental group $\pi_1(\mathcal{M}_g)$ is generated by Dehn twists $(\tau_1, \ldots, \tau_n)$ with

$$[\tau_i, \tau_{i+1}] = 1.$$ 

For a negatively curved manifold this implies $\pi_1$ is virtually abelian, a contradiction. 

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Metrics on Teichmüller space

<table>
<thead>
<tr>
<th>Teichmüller metric</th>
<th>Weil-Petersson metric</th>
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</thead>
<tbody>
<tr>
<td>• Complete</td>
<td>• Kähler</td>
</tr>
<tr>
<td>• $\text{vol}(\mathcal{M}_g) &lt; \infty$</td>
<td>• Convex</td>
</tr>
<tr>
<td>• Curvature $-1$ when $g = 1$</td>
<td>• Curvature $\leq 0$</td>
</tr>
<tr>
<td>• Not Riemannian $(g &gt; 1)$</td>
<td>• Curvature $\to -\infty$</td>
</tr>
<tr>
<td></td>
<td>• Incomplete</td>
</tr>
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</table>
Kähler hyperbolicity

**Theorem** The Teichmüller metric on moduli space $\mathcal{M}_g$ is comparable to a Kähler hyperbolic metric $h$.

$\Rightarrow$ We have a symplectic form $\omega$ on $\mathcal{M}_g$ such that:

- The corresponding Kähler length satisfies:
  $$\|v\|^2_h = \omega(v, Jv) \asymp \|v\|^2_{\text{Teich}};$$

- $\tilde{\omega} = d\theta$ for a bounded 1-form $\theta$ on $\text{Teich}(S)$; and

- $(\mathcal{M}_g, h)$ is complete, with finite volume and bounded curvature.
Consequences of Kähler hyperbolicity

In the Teichmüller metric:

- The least eigenvalue of the Laplacian satisfies
  \[ \lambda_0(\text{Teich}(S)) > 0. \]

- For any compact complex submanifold \( N^{2k} \subset \text{Teich}(S) \),
  \[ \text{vol}(N) \leq C \cdot \text{vol}(\partial N). \]

- \( L^2 \)-cohomology of \( \text{Teich}(S) \) lives in the middle dimension.

- The Euler characteristic satisfies:
  \[ \text{sign } \chi(\mathcal{M}_g) = (-1)^{\text{dim}_C \mathcal{M}_g}. \]
The $1/\ell$ metric

Bers’ embedding:

$$\beta_X : \text{Teich}(S) \rightarrow Q(X) \cong T^*_X \text{Teich}(S) \cong \mathbb{C}^{3g-3}.$$  

**Bounded 1-form on** Teich$(S)$:

$$\theta_Y(X) = \beta_X(Y) \in T^*_X \text{Teich}(S).$$

**Theorem** The Weil-Petersson Kähler form is $d$ (bounded):

$$\omega_{WP} = d(i\theta_Y).$$

**The $1/\ell$ metric:**

$$\omega_{1/\ell} = \omega_{WP} - i\delta \sum_{\ell_\alpha(X) < \epsilon} \partial \bar{\partial} \log \frac{\epsilon}{\ell_\alpha}$$

**Theorem** The $1/\ell$-metric is Kähler hyperbolic, and comparable to the Teichmüller metric.

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**Compare:** The $1/d$ metric.

On a smoothly bounded region $\Omega \subset \mathbb{C}$,

$$\rho_{1/d} = \frac{|dz|}{d(z, \partial \Omega)}$$

is comparable to the hyperbolic metric.
Kepler’s orbs

<table>
<thead>
<tr>
<th></th>
<th>Predicted</th>
<th>Actual</th>
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</thead>
<tbody>
<tr>
<td>Jupiter/Saturn</td>
<td>577</td>
<td>635</td>
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<tr>
<td>Mars/Jupiter</td>
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<td>Earth/Mars</td>
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<td>Venus/Earth</td>
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<td>794</td>
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<tr>
<td>Mercury/Venus</td>
<td>577</td>
<td>723</td>
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Orbs
Next time

Instead of studying maps

\[ S^1 \to \mathcal{M}_g, \]

we will study maps

\[ X \to \mathcal{M}_g \]

where \( X \) is a Riemann surface.