Geometric representation theory of $\mathrm{SL}_2(\mathbb{R})$

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1 Introduction

The representation theory of $\mathrm{SL}_2(\mathbb{R})$ is of interest to number theorists and physicists, notably through its connection to the theory of modular forms. The importance of the discrete series representations where these modular forms live was recogized classically, but a full classification of the irreducible admissible representations was only accomplished by Bargmann in 1947. The goal of this note is to give an algebro-geometric account of the classification using the theory of $\mathscr{D}$-modules.

Notations:

$$\begin{aligned} G_{\mathbb{R}} &= \mathrm{SL}_2(\mathbb{R}) \\ G &= \mathrm{SL}_2(\mathbb{C}) \\ \mathfrak{g} &= \mathfrak{sl}_2(\mathbb{C}) \\ U(\mathfrak{g}) &= \text{ the universal enveloping algebra of } \mathfrak{g} \\ Z(\mathfrak{g}) &= \text{ the center of } U(\mathfrak{g}) \\ K_{\mathbb{R}} &= \mathrm{SO}_2(\mathbb{R}) \cong S^1 \\ K &= \mathrm{SO}_2(\mathbb{C}) \cong \mathbb{G}_m \end{aligned}$$

2 Harish-Chandra modules

We are interested *a priori* in representations of $G_{\mathbb{R}}$ acting on certain topological vector spaces (e.g. Hilbert, Banach, and Fréchet spaces), but this involves functional analysis. If we naïvely attempted to classify irreducible admissible representations up to isomorphism, we would find that many representations which are defined by the same formulas and therefore "the same" representation-theoretically are non-isomorphic for topological reasons.

Example 2.1. Consider the adjoint action of $G_{\mathbb{R}}$ on $\mathbb{P}^1_{\mathbb{R}} \cong S^1$. This is induces an action of $G_{\mathbb{R}}$ on the Hilbert space $L^2(S^1)$, the Banach spaces $L^p(S^1)$ for all p, the Frechet space $C^\infty(S^1)$, etc.

For this reason we will formulate the notion of infinitesimal equivalence, which is insensitive to these analytic subtleties. This leads naturally to the consideration of Harish-Chandra modules.

Recall that a representation of $G_{\mathbb{R}}$ on a topological complex vector space V (we are being intentionally vague about the continuity condition imposed) is called *admissible* provided that any irreducible representation of $K_{\mathbb{R}}$ has finite multiplicity in V. This is a very mild finiteness condition, and irreducible representations which are not admissible should be regarded as pathological. A theorem of Harish-Chandra asserts that any irreducible unitary representation is admissible. We call $V \neq 0$ *irreducible* if it has no closed $G_{\mathbb{R}}$-invariant subspaces other than 0 and V.

Definition 2.2. A $(\mathfrak{g}, K)$*-module* is a complex vector space V with actions of $\mathfrak{g}$ and K which are compatible in the following senses:

(i) the two induced actions of $\mathfrak{k} = \mathrm{Lie}(K)$ agree, and

(ii) the action map $U(\mathfrak{g}) \otimes V \to V$ is K-equivariant.

Morphisms of $(\mathfrak{g}, K)$-modules are required to commute with the actions of $\mathfrak{g}$ and K. A *Harish-Chandra module* is a $(\mathfrak{g}, K)$-module of finite length.

In the previous definition and elsewhere we consider K as an algebraic group, so that representations of K are required to be algebraic and in particular locally finite. Harish-Chandra proved that any Harish-Chandra module is admissible, where the definition of admissible is the same as for $G_{\mathbb{R}}$-representations with $K_{\mathbb{R}}$ replaced by K.

For any representation V of $G_{\mathbb{R}}$, denote by $\mathrm{HC}(V)$ the subspace of vectors on which the action of $K_{\mathbb{R}}$ is locally finite, hence extends to an algebraic action of K. As shown by Harish-Chandra, if V is admissible then $\mathrm{HC}(V)$ is dense in V and inherits a $\mathfrak{g}$-action, and it is not hard to check that this gives $\mathrm{HC}(V)$ the structure of a $(\mathfrak{g}, K)$-module. If V is not only admissible but also of finite length, then $\mathrm{HC}(V)$ is a Harish-Chandra module, and V is irreducible if and only if $\mathrm{HC}(V)$ is an irreducible Harish-Chandra module.

Definition 2.3. Two admissible representations of $G_{\mathbb{R}}$ are called *infinitesimally equivalent* provided that the corresponding $(\mathfrak{g}, K)$-modules are isomorphic.

Summarizing these considerations, we obtain the following result.

Theorem 2.4 (Harish-Chandra)**.** *The assignment* HC *defines a faithful and essentially surjective functor from admissible finite length representations of $G_{\mathbb{R}}$ to Harish-Chandra modules. Moreover, this functor induces a bijection between irreducible admissible representations of $G_{\mathbb{R}}$ up to infinitesimal equivalence and irreducible Harish-Chandra modules up to isomorphism.*

So we have reduced our problem to the classification of irreducible Harish-Chandra modules.

3 The center of the enveloping algebra

The following result is a version of Schur's lemma.

Proposition 3.1. *The action of $Z(\mathfrak{g})$ on an irreducible Harish-Chandra module factors through an infinitesimal character $Z(\mathfrak{g}) \to \mathbb{C}$.*

The infinitesimal character is an important invariant: we will eventually see that there are finitely many irreducible Harish-Chandra modules with a given central character. We will shortly give a concrete description of $Z(\mathfrak{g})$. It is well-known that in our case $\mathfrak{g} = \mathfrak{sl}_2$ the center is generated by the Casimir element, but we will use a method which generalizes to an arbitrary semisimple Lie algebra.

Let $T \subset G$ be the diagonal torus and $\mathfrak{h} \cong \mathbb{C}$ its Lie algebra. Write $W = S_2$ for the Weyl group. Now W acts linearly on $\mathfrak{h}$ in the usual way, but for our purposes here it is more convenient to use the "dotted action" given by

$$w \cdot h = w(h+1) - 1.$$

Thus the involution $\sigma \in W$ reflects $\mathfrak{h}$ over the fixed point -1.

Theorem 3.2 (Harish-Chandra isomorphism)**.** *There is a canonical isomorphism*

$$Z(\mathfrak{g}) \xrightarrow{\sim} \mathrm{Sym}(\mathfrak{h})^{W,\cdot}.$$

Corollary 3.2.1. *There is a canonical isomorphism*

$$\mathfrak{h}^* /\!/ W := \operatorname{Spec} \mathrm{Sym}(\mathfrak{h})^{W,\cdot} \xrightarrow{\sim} \operatorname{Spec} Z(\mathfrak{g}).$$

Taking $\mathbb{C}$-points of both sides in the corollary, we obtain a W-invariant map $\lambda \mapsto \chi_\lambda$ from $\mathfrak{h}^*$ to the set of infinitesimal characters. For any $\lambda \in \mathfrak{h}^* \cong \mathbb{C}$ let $\mathfrak{m}_\lambda = \ker \chi_\lambda$ and write $U(\mathfrak{g})_\lambda := U(\mathfrak{g})/\mathfrak{m}_\lambda U(\mathfrak{g})$ (note that since $\mathfrak{m}_\lambda$ is central, it generates a two-sided ideal in $U(\mathfrak{g})$, so $U(\mathfrak{g})_\lambda$ is an algebra). Denote by $\mathfrak{g}\text{-mod}_\lambda$ the category of $\mathfrak{g}$-modules with infinitesimal character χ_λ, or equivalently $U(\mathfrak{g})_\lambda$-modules. Similarly, write $(\mathfrak{g}, K)\text{-mod}_\lambda$ for the category of $(\mathfrak{g}, K)$-modules whose underlying $\mathfrak{g}$-module has infinitesimal character χ_λ.

The Harish-Chandra isomorphism can be characterized as follows. An element $z \in Z(\mathfrak{g})$ maps to a certain W-invariant polynomial function $f_z : \mathfrak{h}^* \to \mathbb{C}$. Identifying $\mathfrak{h}^* \cong \mathbb{C}$, note that it suffices to specify the values of f_z on the nonnegative integers, which are Zariski dense in $\mathbb{C}$. For each $n \geq 1$ there is a unique n-dimensional irreducible representation V^{n-1} of $\mathfrak{g}$, and Schur's lemma tells us that z acts on V_{n-1} by a scalar. That scalar is $f_z(n-1)$.

In our case $\mathfrak{g} = \mathfrak{sl}_2$, we have $\mathrm{Sym}(\mathfrak{h}) \cong \mathbb{C}[t]$ and $\mathrm{Sym}(\mathfrak{h})^{W,\cdot} \cong \mathbb{C}[t]^\sigma = \mathbb{C}[t^2 + 2t]$. So an infinitesimal character $Z(\mathfrak{g}) \to \mathbb{C}$ is a complex number λ modulo the involution $\lambda \mapsto -\lambda - 2$. In fact, (up to a scalar) $t^2 + 2t$ corresponds to the Casimir element under the Harish-Chandra isomorphism.

4 Algebraic $\mathscr{D}$-modules

Fix a smooth variety X over $\mathbb{C}$. We consider X as an algebraic variety, as opposed to e.g. a complex manifold, so all sheaves on X are for the Zariski topology and $\mathscr{O}_X$ denotes the sheaf of polynomial functions.

Definition 4.1. The *sheaf of differential operators* $\mathscr{D}_X$ on X is the filtered subalgebra of $\mathscr{E}nd_{\mathbb{C}}(\mathscr{O}_X)$ defined inductively as follows:

- $\mathscr{D}_X^0 = \mathscr{O}_X$, and
- $D \in \mathscr{D}_X^{n+1}$ provided that $[D, f] \in \mathscr{D}_X^n$ for any $f \in \mathscr{O}_X$.

In particular $\mathscr{D}_X^1 / \mathscr{D}_X^0 \cong \mathscr{T}_X$, the tangent sheaf of vector fields on X. One can view $\mathscr{T}_X$ as a Lie algebroid on X under the Lie bracket, and then $\mathscr{D}_X$ is its universal enveloping algebra, which is naturally a filtered quasicoherent sheaf of associative algebras on X. Thus the associated graded algebra $\mathrm{gr}\, \mathscr{D}_X \cong \mathrm{Sym}\, \mathscr{T}_X$, so one says that differential operators are a quantization of functions on the cotangent bundle.

Definition 4.2. The category $\mathscr{D}(X)$ of *left $\mathscr{D}$-modules on X* is the abelian category of sheaves of left $\mathscr{D}_X$-modules which are quasicoherent as $\mathscr{O}_X$-modules.

Example 4.3. Suppose that $X = \mathrm{Spec}\, A$ is affine and note that any system of linear partial differential equations with coefficients in A gives rise to a cyclic $\mathscr{D}$-module on X. Indeed, such a system has the form $D_1 f = \cdots = D_r f = 0$ for some $D_1, \cdots, D_r \in \Gamma(X, \mathscr{D}_X)$, and if $\mathscr{I}$ denotes the left ideal sheaf they generate in $\mathscr{D}_X$ put $\mathscr{M} := \mathscr{D}_X / \mathscr{I} \mathscr{D}_X$. Conversely, any cyclic $\mathscr{D}$-module on X comes from such a system of differential equations, which is finite because $\Gamma(X, \mathscr{D}_X)$ is Noetherian [insert reference].

Example 4.4. If $\mathscr{E}$ is a vector bundle on X, then a $\mathscr{D}$-module structure on $\mathscr{E}$ is just a flat connection. A $\mathscr{D}$-module of this form is called *lisse*. In fact, if $\mathscr{F}$ is a $\mathscr{D}$-module which is coherent as an $\mathscr{O}$-module then it is automatically a vector bundle.

Example 4.5. The *exponential $\mathscr{D}$-module* exp on $\mathbb{A}^1$ can be defined formally as $\exp = \mathscr{O}_{\mathbb{A}^1} \cdot e^x$ where $\frac{d}{dx} e^x = e^x$. In other symbols, e^x is the trivial bundle with the connection $\nabla \cdot 1 = dx$.

Denote by $\mathscr{D}(X)^r$ the category of right $\mathscr{D}$-modules. The canonical sheaf ω_X has a natural structure of right $\mathscr{D}$-module: a vector field acts on a top form as minus the Lie derivative. More generally, if $\mathscr{M}$ is a left $\mathscr{D}$-module then $\mathscr{M} \otimes_{\mathscr{O}_X} \omega_X$ has a right $\mathscr{D}$-module structure with the action of vector fields given by

$$(m \otimes \omega) \cdot \xi = m \otimes (\omega \cdot \xi) - (\xi \cdot m) \otimes \omega.$$

This construction induces an equivalence of categories $\mathscr{D}(X) \tilde{\to} \mathscr{D}(X)^r$, with inverse defined on the underlying quasicoherent sheaves by

$$\mathscr{M} \mapsto \omega_X^{-1} \otimes_{\mathscr{O}_X} \mathscr{M} = \mathscr{H}om_{\mathscr{O}_X}(\omega_X, \mathscr{M}).$$

The left action of vector fields on $\omega_X^{-1} \otimes_{\mathscr{O}_X} \mathscr{M}$ is given by

$$\xi \cdot (s \otimes m) = (\xi \cdot s) \otimes m + s \otimes (\xi \cdot m).$$

Unfortunately it is hard to develop the general theory further without using some homological algebra. It will therefore be useful to know that $\mathscr{D}(X)$ has enough injective and locally projective objects [insert reference].

Call a $\mathscr{D}$-module *$\mathscr{D}$-coherent* if it is locally finitely generated over $\mathscr{D}_X$. One can check that if $\mathscr{M}$ is $\mathscr{D}$-coherent then $\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$ is quasicoherent. We will need the right derived functor $R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{N})$, which can be computed by choosing an injective resolution of $\mathscr{N}$ or a locally projective resolution of $\mathscr{M}$ by $\mathscr{D}$-coherent modules.

For a $\mathscr{D}$-coherent left module $\mathscr{M}$, the complex $R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)$ has a structure of right $\mathscr{D}$-module induced by right multiplication on $\mathscr{D}_X$. The *dual* of $\mathscr{M}$ is the complex of left $\mathscr{D}$-modules

$$\mathbb{D}\mathscr{M} := \omega_X^{-1} \otimes_{\mathscr{O}_X} R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \mathscr{D}_X)[\dim X].$$

We also have $\mathbb{D}\mathscr{M} = R\mathscr{H}om_{\mathscr{D}_X}(\mathscr{M}, \omega_X^{-1} \otimes_{\mathscr{O}_X} \mathscr{D}_X)[\dim X]$, where we are using the fact that $\omega_X^{-1} \otimes_{\mathscr{O}_X} \mathscr{D}_X$ has two commuting left $\mathscr{D}$-module structures.

Definition 4.6. Call a $\mathscr{D}$-module $\mathscr{M}$ *holonomic* if it is $\mathscr{D}$-coherent and $H^n(\mathbb{D}\mathscr{M}) = 0$ for $n \neq 0$. Write $\mathscr{D}_{\mathrm{hol}}(X)$ for the full subcategory consisting of holonomic $\mathscr{D}$-modules.

That is, the dual of a holonomic $\mathscr{D}$-module is again a $\mathscr{D}$-module, as opposed to a complex. This explains the cohomological shift in the definition of $\mathbb{D}$. There is a functorial isomorphism $\mathscr{M} \tilde{\to} \mathbb{D}^2\mathscr{M}$ for holonomic $\mathscr{M}$, so $\mathbb{D}$ is a contravariant autoequivalence of $\mathscr{D}_{\mathrm{hol}}(X)$.

Example 4.7. If $\mathscr{E}$ is lisse, then $\mathbb{D}\mathscr{E}$ is the dual bundle $\mathscr{E}^\vee$ with the dual connection. In particular $\mathscr{E}$ is holonomic.

Let $f : X \to Y$ be a morphism of smooth varieties. In order to define pullback and pushforwards along f, it will be useful to introduce the *transfer bimodules*. Let

$$\mathscr{D}_{X\to Y} := \mathscr{O}_X \otimes_{f^\bullet \mathscr{O}_Y} f^\bullet \mathscr{D}_Y$$

be the pullback of $\mathscr{D}_Y$ as a quasicoherent sheaf, where $f^\bullet$ is the inverse image of sheaves. Then $\mathscr{D}_{X\to Y}$ can be given the structure of a $(\mathscr{D}_X, f^\bullet\mathscr{D}_Y)$-bimodule as follows. The action of $f^\bullet\mathscr{D}_Y$ is by right multiplication. As for the left $\mathscr{D}_X$-action, write $\mathrm{d}f : \mathscr{T}_X \to \mathscr{O}_X \otimes_{f^\bullet\mathscr{O}_Y} f^\bullet\mathscr{T}_Y$ for the differential of f and set

$$\xi \cdot (g \otimes D) = (\xi \cdot g) \otimes D + g(\mathrm{d}f)(\xi)D,$$

where ξ is a vector field on X and $(\mathrm{d}f)(\xi)D$ makes sense in $\mathscr{O}_X \otimes_{f^\bullet\mathscr{O}_Y} f^\bullet\mathscr{D}_Y$. We can also produce a $(f^\bullet\mathscr{D}_Y, \mathscr{D}_X)$-bimodule

$$\mathscr{D}_{Y\leftarrow X} := f^\bullet\omega_Y^{-1} \otimes_{f^\bullet\mathscr{O}_Y} \mathscr{D}_{X\to Y} \otimes_{\mathscr{O}_X} \omega_X$$

from $\mathscr{D}_{X\to Y}$.

Now given a left $\mathscr{D}$-module $\mathscr{M}$ on Y, we define a complex of left $\mathscr{D}$-modules on X by

$$f^\Delta\mathscr{M} = \mathscr{D}_{X\to Y} \overset{L}{\otimes}_{f^\bullet\mathscr{D}_Y} f^\bullet\mathscr{M}.$$

Let $f^! = f^\Delta[\dim X - \dim Y]$.

If f is smooth, then for any $\mathscr{D}$-module $\mathscr{M}$ on Y we have $H^n(f^\Delta\mathscr{M})$ for all $n \neq 0$. Thus there is an exact *normalized pullback functor* $f^\Delta : \mathscr{D}(Y) \to \mathscr{D}(X)$.

Given a left $\mathscr{D}$-module $\mathscr{M}$ on X, we define

$$f_*\mathscr{M} := Rf_\bullet(\mathscr{D}_{Y\leftarrow X} \overset{L}{\otimes}_{\mathscr{D}_X} \mathscr{M}),$$

where $f_\bullet$ is the pushforward of sheaves. Note that $f_*\mathscr{M}$ has a left $\mathscr{D}_Y$-action through the map $\mathscr{D}_Y \to f_\bullet f^\bullet\mathscr{D}_Y$.

If f is a closed embeddding then $f^!$ has another description which is more easily given for right modules. Define $\Gamma_X : \mathscr{D}(Y) \to \mathscr{D}(X)$ on the underlying quasicoherent sheaves so that $\Gamma_X(U, \mathscr{M}) \subset \Gamma(U, \mathscr{M})$ consists of sections annihilated by the ideal sheaf of X. If ξ is a vector field on X, let $\tilde{\xi}$ be an extension to Y which is tangent to X. Then we have $m \cdot \xi := m \cdot \tilde{\xi} \in \Gamma_X(U, \mathscr{M})$, and this does not depend on the choice of $\tilde{\xi}$. Since Γ_X is left exact we consider its right derived functor $R\Gamma_X$, and there is a functorial isomorphism

$$f^!\mathscr{M} \cong \omega_X^{-1} \otimes_{\mathscr{O}_X} R\Gamma_X(\mathscr{M} \otimes_{\mathscr{O}_X} \omega_X).$$

Theorem 4.8 (Kashiwara)**.** *If f is a closed embedding then f_* and $f^!$ define mutually inverse equivalences between $\mathscr{D}(X)$ and $\mathscr{D}_X(Y)$, where $\mathscr{D}_X(Y)$ denotes the full subcategory of $\mathscr{D}$-modules on Y whose underlying quasicoherent sheaf is set-theoretically supported on X.*

Theorem 4.9. *The functors $f^!$ and f_* preserve holonomicity.*

We let $f_!$ and f^* be the left adjoints of $f^!$ and f_* respectively at the level of derived categories, which for a general f are only partially defined. However, if we restrict to holonomic $\mathscr{D}$-modules then $f_! = \mathbb{D} f_* \mathbb{D}$ and $f^* = \mathbb{D} f^! \mathbb{D}$. For f proper we have $f_! = f_*$. When f is smooth $f^* = f^![2(\dim Y - \dim X)]$.

Example 4.10. Suppose that Y is a point. Then $f^!\mathbb{C} = \mathscr{O}_X[\dim X]$ and $f^*\mathbb{C} = \mathscr{O}_X[-\dim X]$. Moreover $f_* f^* \mathbb{C}$ computes the de Rham cohomology of X, and $f_! f^* \mathbb{C}$ computes the compactly supported de Rham cohomology (this is essentially Poincaré duality).

Now we can give some alternative characterizations of holonomic $\mathscr{D}$-modules.

Theorem 4.11. *For a $\mathscr{D}$-coherent module $\mathscr{M}$ on X the following are equivalent:*

- *(i) $\mathscr{M}$ is holonomic,*
- *(ii) for any $x \in X$ the fiber $i_x^! \mathscr{M}$ has finite-dimensional cohomology,*
- *(iii) X admits a stratification $X = \sqcup X_k$ into locally closed subvarieties such that $j_k^! \mathscr{M}$ is $\mathscr{O}$-coherent, i.e. a vector bundle with connection, for each stratum $j_k : X_k \to X$.*

Ironically, none of the characterizations of holonomic $\mathscr{D}$-modules we have given are the standard definition, which uses the notion of singular support. We will not need it in these notes.

5 Extension over a closed subvariety

Now let $i : Y \to X$ be a closed embedding of smooth varieties and $j : U \to X$ the complementary open embedding. For any $\mathscr{D}$-module $\mathscr{M}$ on X there is an exact triangle

$$i_! i^! \mathscr{M} \to \mathscr{M} \to j_* j^* \mathscr{M},$$

and if $\mathscr{M}$ is holonomic we also have

$$j_! j^! \mathscr{M} \to \mathscr{M} \to i_* i^* \mathscr{M}.$$

Notice that here $i_! = i_*$ and $j^* = j^!$.

Note that for any holonomic $\mathscr{D}$-module $\mathscr{N}$ on U, we have $j^* j_! \mathscr{N} = j^! j_! \mathscr{N} = \mathscr{N}$ and therefore by adjunction there is a canonical morphism $j_! \mathscr{N} \to j_* \mathscr{N}$.

If j is affine, then $j_! \mathscr{N}$ and $j_* \mathscr{N}$ are concentrated in degree 0. In this situation we define the *minimal extension functor* $j_{!*} : \mathscr{D}_{\mathrm{hol}}(U) \to \mathscr{D}_{\mathrm{hol}}(X)$ by

$$j_{!*} \mathscr{N} := \mathrm{im}(j_! \mathscr{N} \to j_* \mathscr{N}).$$

Proposition 5.1. *The complexes $i^! j_{!*} \mathscr{N}$ and $i^* j_{!*} \mathscr{N}$ are concentrated in degrees 1 and -1 respectively, and there is an exact sequence*

$$0 \to i_* i^* j_{!*} \mathscr{N}[-1] \to j_! \mathscr{N} \to j_* \mathscr{N} \to i_! i^! j_{!*} \mathscr{N}[1] \to 0.$$

Proof. The standard exact triangle

$$j_! \mathscr{N} = j_! j^! j_{!*} \mathscr{N} \longrightarrow j_{!*} \mathscr{N} \longrightarrow i_* i^* j_{!*} \mathscr{N}$$

rotates to become

$$i_* i^* j_{!*} \mathscr{N}[-1] \longrightarrow j_! \mathscr{N} \longrightarrow j_{!*} \mathscr{N}.$$

Since $j_! \mathscr{N} \to j_{!*} \mathscr{N}$ is an epimorphism, $i_* i^* j_{!*} \mathscr{N}[-1]$ is concentrated in degree 0 as claimed. The other half of the exact sequence is similar.

□

Said differently, $i_*i^*j_{!*}\mathscr{N}[-1]$ is the kernel of the epimorphism $j_!\mathscr{N} \to j_{!*}\mathscr{N}$ and $i_!i^!j_!\mathscr{N}[1]$ is the cokernel of the monomorphism $j_{!*}\mathscr{N} \to j_*\mathscr{N}$.

Proposition 5.2. *If $\mathscr{N}$ is a simple $\mathscr{D}$-module on U, then $j_{!*}\mathscr{N}$ is up to isomorphism the unique simple $\mathscr{D}$-module on X which restricts to $\mathscr{N}$. Moreover, $j_{!*}\mathscr{N}$ is the unique simple quotient of $j_!\mathscr{N}$ and the unique simple subsheaf of $j_*\mathscr{N}$.*

Proof. First we prove that $j_{!*}\mathscr{N}$ is simple, so consider a short exact sequence

$$0 \longrightarrow \mathscr{M} \longrightarrow j_{!*}\mathscr{N} \longrightarrow \mathscr{P} \longrightarrow 0.$$

Because $\mathscr{N}$ is simple, the exactness of

$$0 \longrightarrow j^*\mathscr{M} \longrightarrow \mathscr{N} \longrightarrow j^*\mathscr{P} \longrightarrow 0$$

implies that either $j^*\mathscr{M}$ or $j^*\mathscr{P}$ is zero. Suppose $j^*\mathscr{M} = 0$; the other argument is dual.

The composition $\mathscr{M} \to j_{!*}\mathscr{N} \to j_*\mathscr{N}$ must vanish, since

$$\operatorname{Hom}(\mathscr{M}, j_*\mathscr{N}) = \operatorname{Hom}(j^*\mathscr{M}, \mathscr{N}) = 0.$$

Both maps are injective, so $\mathscr{M} = 0$.

Next we prove that $j_{!*}\mathscr{N}$ is the unique simple subsheaf of $j_*\mathscr{N}$. If $\mathscr{M}$ is a simple subsheaf of $j_*\mathscr{N}$, then we obtain a nonzero morphism $j^*\mathscr{M} \to \mathscr{N}$, whence $j^!\mathscr{M} = j^*\mathscr{M} \neq 0$. It follows from the usual exact triangle that $i^*\mathscr{M} = 0$. Since the composition

$$\mathscr{M} \longrightarrow j_*\mathscr{N} \longrightarrow i_!i^!j_{!*}\mathscr{N}[1]$$

corresponds to a morphism $i^*\mathscr{M} \to i^!j_{!*}\mathscr{N}[1]$ it vanishes, so $\mathscr{M} = j_{!*}\mathscr{N}$.

Suppose that $\mathscr{M}$ is a simple $\mathscr{D}$-module on X such that $j^*\mathscr{M} \xrightarrow{\sim} \mathscr{N}$. We immediately obtain an injection $\mathscr{M} \to j_*\mathscr{N}$, which must be an isomorphism onto the unique simple subsheaf $j_{!*}\mathscr{N}$.

□

The following definition describes the simplest extension situation. When a complex is acyclic (i.e. has vanishing cohomology in all degrees) we will abusively write that that complex is equal to 0.

Definition 5.3. We say a holonomic $\mathscr{D}$-module $\mathscr{N}$ on U *extends cleanly to* X if the following equivalent conditions are satisfied:

- $j_!\mathscr{N} \xrightarrow{\sim} j_{!*}\mathscr{N} \xrightarrow{\sim} j_*\mathscr{N}$,
- $i^*j_*\mathscr{N} = 0$,
- $i^!j_!\mathscr{N} = 0$,
- $i^*j_{!*}\mathscr{N} = 0 = i^!j_{!*}\mathscr{N}$.

If $\mathscr{N}$ is simple then these conditions are equivalent to $j_!\mathscr{N}$ and $j_*\mathscr{N}$ being simple.

Example 5.4. Put $X = \mathbb{A}^1$, $Y = \{0\}$, and $U = \mathbb{A}^1 \setminus \{0\}$. Let's study extensions of the $\mathscr{D}$-module $\mathscr{O}_U$. We have $j_{!*}\mathscr{O}_U \cong \mathscr{O}_X$ because $\mathscr{O}_X$ is the simple $\mathscr{D}$-module on X which extends $\mathscr{O}_U$. Since $i^{\Delta}\mathscr{O}_X$ agrees with the usual left derived pullback of $\mathscr{O}_X$ as a quasicoherent sheaf we have $i^{\Delta}\mathscr{O}_X = \mathbb{C}$, so that $i^!\mathscr{O}_X = i^{\Delta}\mathscr{O}_X[-1] = \mathbb{C}[-1]$ and

$$i^*\mathscr{O}_X = \mathbb{D}i^!\mathbb{D}\mathscr{O}_X = \mathbb{D}(\mathbb{C}[-1]) = \mathbb{C}[1].$$

In particular, $\mathscr{O}_U$ does not extend cleanly to X. Note that $i^!\mathscr{O}_X[1] = \mathbb{C} = i^*\mathscr{O}_X[-1]$ are concentrated in degree 0 as expected, and let $\delta_0 = i_*\mathbb{C}$ be the *delta sheaf at* 0. Proposition 5.1 yields a short exact sequence

$$0 \to \delta_0 \to j_!\mathscr{O}_U \to \mathscr{O}_X \to 0$$

and its dual

$$0 \to \mathscr{O}_X \to j_*\mathscr{O}_U \to \delta_0 \to 0.$$

Observe that the first sequence is nonsplit because

$$\operatorname{Hom}(j_!\mathscr{O}_U, \delta_0) = \operatorname{Hom}(\mathscr{O}_U, j^!\delta_0) = 0,$$

and similarly for the second sequence.

Clean extensions are a bit harder to construct but very useful when they occur. We will see an example in the next section.

6 Equivariant $\mathscr{D}$-modules

Let H be an algebraic group with operation $\mu : H \times H \to H$. A *(rank 1) character sheaf*, sometimes called a *multiplicative local system*, on H is a rank 1 local system (i.e. line bundle with connection) $\mathscr{L} \in \mathscr{D}(H)$ together with an isomorphism $\mu^{\Delta}\mathscr{L} \tilde{\to} \mathscr{L} \boxtimes \mathscr{L}$ satisfying an associativity condition on $H \times H \times H$. For example, there is the *trivial character sheaf* $\mathscr{L} = \mathscr{O}_H$. The automorphism group of any character sheaf on H is $\operatorname{Hom}(\pi_0(H), \mathbb{G}_m)$.

If H acts on a variety X with action map $\alpha : H \times X \to X$, one can form the abelian category $\mathscr{D}(X)^{\mathscr{L}}$ of *$\mathscr{L}$-equivariant $\mathscr{D}$-modules on X*, an object of which is a $\mathscr{D}$-module $\mathscr{F}$ on X together with an isomorphism $\alpha^{\Delta}\mathscr{F} \tilde{\to} \mathscr{L} \boxtimes \mathscr{F}$ satisfying a cocycle condition on $H \times H \times X$. When $\mathscr{L} = \mathscr{O}_H$ is the trivial character sheaf, $\mathscr{D}(X)^H := \mathscr{D}(X)^{\mathscr{O}_H}$ is the category of *(strongly) H-equivariant $\mathscr{D}$-modules on X*.

Proposition 6.1. *Suppose that H acts transitively on X. If the restriction of $\mathscr{L}$ to the identity component H_x^0 of the stabilizer of some (equivalently, any) point $x \in X$ is nontrivial, then $\mathscr{D}(X)^{\mathscr{L}} = 0$. Given an isomorphism of character sheaves $\mathscr{O}_{H_x^0} \tilde{\to} \mathscr{L}|_{H_x^0}$, one obtains an equivalence of categories*

$$\mathscr{D}(X)^H \xrightarrow{\sim} \mathscr{D}(\bullet)^{H_x} = \operatorname{Rep}(\pi_0(H_x)).$$

In particular, the proof of this proposition shows that when H acts transitively on X, any holonomic object of $\mathscr{D}(X)^{\mathscr{L}}$ is lisse. In general, the !-restriction of a holonomic $\mathscr{L}$-equivariant $\mathscr{D}$-module to any H-orbit is lisse.

Example 6.2. We claim that exp extends cleanly along $j : \mathbb{A}^1 \to \mathbb{P}^1$. Let $\mathbb{G}_a$ act by translations on $\mathbb{P}^1$, so j is $\mathbb{G}_a$-equivariant and ∞ is the fixed point. Since exp is a character sheaf on $\mathbb{G}_a$, it also has the structure of an exp-equivariant sheaf. So do $j_* \exp$ and $i^* j_* \exp$, where $i : \{\infty\} \to \mathbb{P}^1$. Proposition 6.1 implies that $\mathscr{D}(\infty)^{\exp} = 0$ because exp is a nontrivial character sheaf, so $i^* j_* \exp = 0$ as desired.

Proposition 6.3. *Suppose H has finitely many orbits in X. If $\mathscr{M}$ is a simple holonomic object of $\mathscr{D}(X)^{\mathscr{L}}$, then there is an H-orbit $Q \subset X$ and a simple holonomic object $\mathscr{N} \in \mathscr{D}(X)^{\mathscr{L}}$ with an $\mathscr{L}$-equivariant isomorphism $\mathscr{M} \tilde{\to} j_{!*}\mathscr{N}$.*

7 Twisted $\mathscr{D}$-modules and Beilinson-Bernstein localization

Denote by $B \subset G$ the subgroup of upper triangular matrices and $N \cong \mathbb{G}_a$ its subgroup with diagonal entries equal to 1. In this section we relate the representation theory of $\mathfrak{g}$ to the geometry of the *flag variety* $G/B = \mathbb{P}^1$ and the *base affine space* $G/N = \mathbb{A}^2 \setminus \{0\}$.

We can already state the localization theorem for representations with the trivial infinitesimal character $\chi_0 : Z(\mathfrak{g}) \to \mathbb{C}$, which is just the restriction to $Z(\mathfrak{g})$ of the augmentation homomorphism $U(\mathfrak{g}) \to \mathbb{C}$. These correspond to "untwisted" $\mathscr{D}$-modules.

The action of G on G/B differentiates to yield a Lie algebra homomorphism $\mathfrak{g} \to \Gamma(G/B, \mathscr{T}_{G/B})$, which induces a homomorphism of associative algebras $U(\mathfrak{g}) \to \Gamma(G/B, \mathscr{D}_{G/B})$.

Theorem 7.1 (Beilinson-Bernstein localization)**.** *The homomorphism constructed above induces an isomorphism* $U(\mathfrak{g})_0 \tilde{\rightarrow} \Gamma(G/B, \mathscr{D}_{G/B})$*, and the resulting functor*

$$\Gamma : \mathscr{D}(G/B) \longrightarrow U(\mathfrak{g})_0\text{-mod}$$

is an equivalence of categories.

Thus $\mathscr{D}$-modules on G/B are the same as $\mathfrak{g}$-modules with trivial infinitesimal character.

In order to get representations with nontrivial central character, we must work with twisted $\mathscr{D}$-modules on G/B. There are several ways to approach the latter, but in any case to each $\lambda \in \mathfrak{h}^* \cong \mathbb{C}$ there is attached an abelian category $\mathscr{D}^\lambda(G/B)$, so that $\mathscr{D}^0(G/B) = \mathscr{D}(G/B)$ and for any $n \in \mathbb{Z}$ there is a canonical equivalence

$$\mathscr{D}^\lambda(G/B) \tilde{\longrightarrow} \mathscr{D}^{\lambda+n}(G/B).$$

The author's preferred approach is to think of twisted $\mathscr{D}$-modules as certain equivariant sheaves on the base affine space G/N. Let T be the diagonal torus in G, which we identify with $\mathbb{G}_m$ via the map

$$t \mapsto \begin{bmatrix} t & 0 \\ 0 & t^{-1} \end{bmatrix}.$$

Define the $\mathscr{D}$-module x^λ on T to be $\mathscr{O}_T$ as an $\mathscr{O}_T$-module, with the connection defined by

$$\partial \cdot 1 = \frac{\lambda}{x} \cdot 1.$$

It is not hard to check that x^λ is a character sheaf on T. Notice that x^n is canonically trivial for all $n \in \mathbb{Z}$, and that $x^\lambda \cong x^\mu$ if and only if $\lambda - \mu \in \mathbb{Z}$.

Since T normalizes N, it also acts on G/N by right translations, which makes $G/N \to G/B$ into a T-bundle. Concretely, this is just the dilation action which makes $\mathbb{A}^2 \setminus \{0\}$ a $\mathbb{G}_m$-bundle over $\mathbb{P}^1$.

Definition 7.2. The category of λ*-twisted* $\mathscr{D}$*-modules on* G/B is defined to be

$$\mathscr{D}^\lambda(G/B) := \mathscr{D}(G/N)^{x^\lambda}.$$

The aforementioned T-action on G/N differentiates to give a map $\varphi : \mathfrak{h} \to \Gamma(G/N, \mathscr{D}_{G/N})$. Consider the left ideal $\mathscr{I}_\lambda$ of $\mathscr{D}_{G/N}$ generated by the sections $\varphi(h) - \lambda(h)$ for all $h \in \mathfrak{h}$. One checks that $\mathscr{I}_\lambda$ and hence $\mathscr{D}_{G/N}/\mathscr{I}_\lambda$ are x^λ-equivariant.

Definition 7.3. The *sheaf of twisted differential operators*, or *TDO* for short, corresponding to λ is

$$\mathscr{D}_\lambda := (\mathscr{D}_{G/N}/\mathscr{I}_\lambda)^T \in \mathscr{D}(G/B).$$

In fact $\mathscr{D}_\lambda$ is a filtered quasicoherent sheaf of associative algebras with the property that $\operatorname{gr} \mathscr{D}_\lambda \cong \operatorname{Sym} \mathscr{T}_X$, so we have constructed a family of quantizations of functions on the cotangent bundle parameterized by weights $\lambda \in \mathfrak{h}^*$. In particular we have $\mathscr{D}_0 = \mathscr{D}_{G/B}$. Let $\mathscr{D}_\lambda$-mod denote the abelian category of sheaves of $\mathscr{D}_\lambda$-modules on G/B, quasicoherent as $\mathscr{O}_{G/B}$-modules.

Proposition 7.4. *The functor* $\mathscr{F} \mapsto \mathscr{F}^T$ *defines an equivalence*

$$\mathscr{D}^\lambda(G/B) \tilde{\longrightarrow} \mathscr{D}_\lambda\text{-mod}.$$

The action of G on G/N by left translations induces a homomorphism $U(\mathfrak{g}) \to \Gamma(G/N, \mathscr{D}_{G/N})$. Composing with the projection yields $U(\mathfrak{g}) \to \Gamma(G/N, \mathscr{D}_{G/N}/\mathscr{I}_\lambda)$, which lands in the T-invariants because the left G action commutes with the right T-action. The map $U(\mathfrak{g}) \to \Gamma(G/B, \mathscr{D}_\lambda)$ so obtained is in fact an algebra homomorphism.

Call a weight λ *dominant* if $\lambda \neq -2, -3, \cdots$. We say λ is *dominant regular* if λ is dominant and $\lambda \neq -1$.

Theorem 7.5 (Twisted Beilinson-Bernstein localization)**.** *The homomorphism* $U(\mathfrak{g}) \to \Gamma(G/B, \mathscr{D}_\lambda)$ *constructed above induces an isomorphism* $U(\mathfrak{g})_\lambda \tilde{\to} \Gamma(G/B, \mathscr{D}_\lambda)$*, and the functor*

$$\Gamma_\lambda : \mathscr{D}^\lambda(G/B) \longrightarrow U(\mathfrak{g})_\lambda\text{-}mod$$

defined by $\Gamma_\lambda(\mathscr{F}) := \Gamma(G/N, \mathscr{F})^T$ *is an equivalence of categories for all dominant regular* λ*. The functor* Γ_{-1} *is exact and essentially surjective.*

Notice that under the equivalence of Proposition 7.4, the functor Γ_λ just takes global sections of the underlying quasicoherent sheaf on G/B.

8 Twisted Harish-Chandra sheaves

Since the actions of K and T on G/N commute they induce an action of $K \times T$. Thus we can form the category $\mathscr{D}^\lambda(G/B)^K := \mathscr{D}(G/N)^{\mathscr{O}_K \boxtimes z^\lambda}$ of K-equivariant λ-twisted $\mathscr{D}$-modules on G/B, which are often called *(twisted) Harish-Chandra sheaves.* The functor Γ_λ lifts to $\Gamma_\lambda^K : \mathscr{D}_\lambda(G/B)^K \to (\mathfrak{g}, K)\text{-mod}_\lambda$, and Theorem 7.5 immediately implies that Γ_λ^K is an equivalence for all dominant regular λ, and that Γ_{-1}^K is exact and essentially surjective.

Letting $\pi : G/N \to G/B$ denote the projection, we can define $\mathscr{D}^\lambda(Z)^K := \mathscr{D}(\pi^{-1}(Z))^{\mathscr{O}_K \boxtimes z^\lambda}$ for any K-stable subvariety $Z \subset G/B$. If $j : Z \to W$ is an inclusion of K-stable subvarieties, we abusively write $j : \pi^{-1}(Z) \to \pi^{-1}(W)$ as well.

Fix a simple λ-twisted Harish-Chandra sheaf $\mathscr{M}$. By Proposition 6.3, there is an orbit $j : Q \to G/B$, a simple object $\mathscr{N} \in \mathscr{D}^\lambda(Q)^K$, and an isomorphism of Harish-Chandra sheaves $\mathscr{M} \tilde{\to} j_{!*}\mathscr{N}$.

In order to describe the categories $\mathscr{D}^\lambda(Q)^K$, it will be necessary to identify $\mathfrak{k} \tilde{\to} \mathfrak{h}$ so that λ can be viewed as an element of $\mathfrak{k}^*$. A general theorem asserts that maximal tori in reductive groups over an algebraically closed field are conjugate, so in fact the subgroups K and T are conjugate in G. Of course there is a more elementary proof, which goes as follows. The element

$$\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

of $\mathfrak{k}$ has eigenvalues 1 and -1, so it is conjugate by a matrix $g \in \mathrm{GL}_2(\mathbb{C})$ to

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

which belongs to $\mathfrak{h}$. Then rescale so that $g \in \mathrm{SL}_2(\mathbb{C})$.

There are only two isomorphisms $K \tilde{\to} T$, both of which can be obtained by conjugation in G. One can make the isomorphism canonical by choosing positive roots for K and T. But independently of this choice, the induced isomorphism $\mathfrak{k} \tilde{\to} \mathfrak{h}$ preserves the integral weights, so $\lambda \in \mathfrak{h}^*$ is integral if and only if the corresponding element of $\mathfrak{k}^*$ is.

Choose a K-orbit $j : Q \to G/B$ and fix $x \in Q$. Write K_x for the stabilizer of x in K.

Proposition 8.1. *We have* $\mathscr{D}^\lambda(Q)^K = 0$ *unless* λ *integrates to a character of the identity component* K_x^0*. If this is the case then*

$$\mathscr{D}^\lambda(Q)^K \tilde{\longrightarrow} \mathrm{Rep}(K_x/K_x^0).$$

So our classification problem is reduced to computing the stabilizers for K acting on G/B and classifying the irreducible representations of their (finite) component groups.

Since the chosen isomorphism $K \tilde{\to} T$ is induced by conjugation, the actions of K and T by left translation on G/N are isomorphic. Explicitly, the left translation action of $T \cong \mathbb{G}_m$ on $G/N \cong \mathbb{A}^2 \setminus \{0\}$ is given by $t \cdot (x, y) = (tx, t^{-1}y)$. The right action of T (given by $(x, y) \cdot t = (tx, ty)$) commutes with left translation, so we can replace the role of K by that of T acting by left translations. This leaves the situation unchanged, except that many formulas become simpler. Alternatively, one can work with $\mathrm{SU}(1, 1)$ instead of $\mathrm{SL}_2(\mathbb{R})$ since these groups are isomorphic, but the former has compact diagonal torus.

The action of $T \cong \mathbb{G}_m$ by left translations on $G/B \cong \mathbb{P}^1$ is the (inverse?) square of the usual dilation action, so there are three orbits: the two fixed points 0 and ∞ and the open orbit $U = \mathbb{P}^1 \setminus \{0, \infty\}$. We denote by j the inclusion of U into $\mathbb{P}^1$, and i_0 and i_∞ the inclusions of the fixed points. As mentioned above, we will use the same symbols to denote the maps obtained from these by taking preimages along π.

The stabilizer of the fixed points is all of T, so $\mathscr{D}^\lambda(0)^T = 0 = \mathscr{D}^\lambda(\infty)^T$ for $\lambda \notin \mathbb{Z}$. For $\lambda \in \mathbb{Z}$ we have

$$\mathscr{D}^\lambda(0)^T \xrightarrow{\sim} \operatorname{Rep}(T/T^0) = \mathrm{Vect}$$

and likewise for $\mathscr{D}^\lambda(\infty)^T$, so in that case each of these categories has a unique simple object. Since the character sheaf z^λ is (canonically) trivial, the simple object is just $\mathscr{O}$.

As for the open orbit U, the stabilizer of any point is $\mu_2 = \{\pm 1\}$, so the condition that the restriction of λ to $\mu_2^0 = 1$ be integral is vacuous. Thus for all λ we have

$$\mathscr{D}^\lambda(U)^T \xrightarrow{\sim} \operatorname{Rep}(\mu_2),$$

so each of these categories has two simple objects τ_+^λ and τ_-^λ. This notation means that τ_+^λ corresponds to the trivial character $\mu_2 \to \mathbb{G}_m$ and τ_-^λ to the the nontrivial character. Note that

$$\pi^{-1}(U) = \mathbb{G}_m \times \mathbb{G}_m \subset \mathbb{A}^2 \setminus \{0\},$$

and in fact $\tau_+^\lambda = z^{\frac{\lambda}{2}} \boxtimes z^{\frac{\lambda}{2}}$ and $\tau_-^\lambda = z^{\frac{\lambda+1}{2}} \boxtimes z^{\frac{\lambda+1}{2}}$. Indeed, both of these sheaves are simple (they have underlying $\mathscr{O}$-module $\mathscr{O}$) and z^λ-equivariant, but the equivariance datum for τ_-^λ involves the isomorphism $z^\lambda \cong z^{\lambda+1}$. This isomorphism is the $\mathscr{O}$-linear automorphism $1 \mapsto z$ of $\mathscr{O}_{\mathbb{G}_m}$, which acts by -1 on the fiber over $-1 \in \mathbb{G}_m$.

In view of the results in Section 6, these considerations immediately yield the following classification.

Proposition 8.2 (Classification of simple Harish-Chandra sheaves for $\mathrm{SL}_2(\mathbb{R})$)**.** *For any weight λ, there are two simple Harish-Chandra sheaves $j_{!*}\tau_+^\lambda$ and $j_{!*}\tau_-^\lambda$ which extend simple sheaves on U. For $\lambda \in \mathbb{Z}$ there are two additional simple Harish-Chandra sheaves $\delta_0^\lambda := i_{0,*}z^\lambda$ and $\delta_\infty^\lambda := i_{\infty,*}z^\lambda$ supported on the fixed points. For a fixed weight λ, these sheaves are pairwise nonisomorphic and any simple λ-twisted Harish-Chandra sheaf on G/B is isomorphic to one of them.*

There is more to say about the *standard* and *costandard* sheaves $j_!\tau_\pm^\lambda$ and $j_*\tau_\pm^\lambda$ coming from the open orbit. Both sheaves δ_0^λ and δ_∞^λ are standard, costandard, and simple, being supported on closed orbits. On the other hand, $j_!\tau_\pm^\lambda$ and $j_*\tau_\pm^\lambda$ may or may not be simple, depending on λ and $\pm$.

Lemma 8.3. *For any $\mu \in \mathbb{C}$ consider the $\mathscr{D}$-module $z^\mu \boxtimes z^\mu$ on $\mathbb{G}_m \times \mathbb{G}_m$. This sheaf extends cleanly to $\mathbb{A}^2 \setminus \{0\}$ if and only if $\mu \notin \mathbb{Z}$.*

Proof. If $\mu \in \mathbb{Z}$ then $z^\mu \cong \mathscr{O}_{\mathbb{G}_m}$ as a $\mathscr{D}$-module, so $j_{!*}(z^\mu \boxtimes z^\mu) = \mathscr{O}_{\mathbb{A}^2 \setminus \{0\}}$ and the extension is not clean. On the other hand, if $\mu \notin \mathbb{Z}$ then z^μ is a nontrivial character sheaf. Consider the action of $\mathbb{G}_m \times \mathbb{G}_m$ on $\mathbb{A}^2 \setminus \{0\}$ given by $(s, t) \cdot (x, y) = (sx, ty)$. Since $j_*(z^\mu \boxtimes z^\mu)$ is $z^\mu \boxtimes z^\mu$-equivariant, so is its $*$-restriction to the x- and y-axes. There the stabilizers in $\mathbb{G}_m \times \mathbb{G}_m$ are $1 \times \mathbb{G}_m$ and $\mathbb{G}_m \times 1$ respectively, and since $z^\mu \boxtimes z^\mu$ is nontrivial on both of these we can apply Proposition 6.1.

□

It follows that $\tau_+^\lambda = z^{\frac{\lambda}{2}} \boxtimes z^{\frac{\lambda}{2}}$ (respectively $\tau_-^\lambda = z^{\frac{\lambda+1}{2}} \boxtimes z^{\frac{\lambda+1}{2}}$) extends cleanly if and only if $\lambda \notin 2\mathbb{Z}$ (respectively $\lambda \notin 2\mathbb{Z} + 1$). In these cases the standard, costandard, and simple sheaves coincide.

Now suppose that $\lambda \in 2\mathbb{Z}$, so $j_{!*}\tau_+^\lambda \cong \mathscr{O}_{\mathbb{A}^2 \setminus \{0\}}$ (disregarding equivariance) and in particular the extension is not clean. Since τ_+^λ is z^λ-equivariant, so is $j_{!*}\tau_+^\lambda$, and in particular its underlying coherent sheaf is $\mathbb{G}_m$-equivariant for the diagonal action. It is not hard to see that this equivariance datum corresponds to the grading $\mathscr{O}_{\mathbb{A}^2 \setminus \{0\}}(\lambda)$. In other symbols, there is a $\mathbb{G}_m$-equivariant isomorphism $\pi^*\mathscr{O}_{\mathbb{P}^1}(\lambda) \xrightarrow{\sim} j_{!*}(\tau_+^\lambda)$.

Since $j_{!*}\tau_+^\lambda$ is z^λ-equivariant so is $i_0^\triangle j_{!*}\tau_+^\lambda$, and $i_0^\triangle j_{!*}\tau_+^\lambda \cong \mathscr{O}$ as coherent sheaves so we must have $i_0^\triangle j_{!*}\tau_+^\lambda \cong z^\lambda$. Thus $i_{0,*}i_0^\triangle j_{!*}\tau_+^\lambda \cong \delta_0^\lambda$, and similarly $i_{\infty,*}i_\infty^\triangle j_{!*}\tau_+^\lambda \cong \delta_\infty^\lambda$. Apply Proposition 5.1 to obtain an exact sequence

$$0 \to \delta_0^\lambda \oplus \delta_\infty^\lambda \to j_!\tau_+^\lambda \to j_*\tau_+^\lambda \to \delta_0^\lambda \oplus \delta_\infty^\lambda \to 0.$$

Similarly, if $\lambda \in 2\mathbb{Z}+1$ then there is a $\mathbb{G}_m$-equivariant isomorphism of coherent sheaves $\pi^*\mathscr{O}_{\mathbb{P}^1}(\lambda) \cong j_{!*}\tau_-^\lambda$ and an exact sequence

$$0 \to \delta_0^\lambda \oplus \delta_\infty^\lambda \to j_!\tau_-^\lambda \to j_*\tau_-^\lambda \to \delta_0^\lambda \oplus \delta_\infty^\lambda \to 0.$$

9 The classification of irreducible representations of $SL_2(\mathbb{R})$

Theorem 2.4 reduced our problem to the classification of irreducible Harish-Chandra modules. By Proposition 3.1, an irreducible Harish-Chandra module has an infinitesimal character, i.e. it lies in $(\mathfrak{g}, K)$-mod$_\lambda$ for some $\lambda \in \mathfrak{h}^*$. Since the infinitesimal character χ_λ only depends on the orbit of λ under the dotted W-action, i.e. reflection over -1, we may as well restrict ourselves to dominant weights.

Now by Theorem 7.5, for any dominant regular λ the functor Γ_λ induces a bijection between (isomorphism classes of) irreducible objects in $\mathscr{D}^\lambda(G/B)^K$ and irreducible objects in $(\mathfrak{g}, K)$-mod$_\lambda$. The *singular* case $\lambda = -1$ is slightly more complicated: Γ_{-1} induces a bijection between irreducible objects in $\mathscr{D}^{-1}(G/B)^K$ with nonvanishing global sections and irreducible objects in $(\mathfrak{g}, K)$-mod$_{-1}$.

There are three K-orbits in $G/B \cong \mathbb{P}^1$: two fixed points $\{i\}$ and $\{-i\}$ and the open orbit $U = \mathbb{P}^1 \setminus \{\pm i\}$. If we fix an isomorphism $U \cong \mathbb{G}_m$ then K acts on U by the formula $z \cdot w = z^2 w$, and in particular the stabilizer in K of any point in U is $\mu_2 = \{\pm 1\}$.

Since μ_2 is discrete, any weight λ integrates to a character of μ_2, and it follows that $\mathscr{D}^\lambda(U)^K \cong \mathrm{Rep}(\mu_2)$. In particular, for any λ the category $\mathscr{D}^\lambda(U)^K$ has precisely two simple objects, which give rise to two irreducible Harish-Chandra sheaves by applying $j_{!*}$ (here $j : U \to \mathbb{P}^1$). Thus if λ is dominant regular, there are two irreducible Harish-Chandra modules $P_\lambda^{\pm 1}$ with infinitesimal character χ_λ which “come from the open orbit.” If $\lambda = n \in \mathbb{Z}^{\geq 0}$ then one of the $P_\lambda^{\pm 1}$ has finite dimension $n+1$ (whether it is P_λ^{+1} or P_λ^{-1} depends on the parity of n). The infinite-dimensional $P_\lambda^{\pm 1}$ correspond to the *irreducible principal series representations* of $G_\mathbb{R}$. When $\lambda = -1$ both of the resulting Harish-Chandra sheaves have underlying $\mathscr{O}$-module $\mathscr{O}(-1)$, hence have no cohomology.

As for the closed orbits, for either $\{i\}$ or $\{-i\}$ a weight λ integrates to a character of the stabilizer K if and only if $\lambda = n \in \mathbb{Z}$, in which case there are two irreducible Harish-Chandra sheaves which “come from the fixed points.” When $n \geq -1$ we denote the corresponding Harish-Chandra modules by $D_n^\pm$. The modules $D_n^\pm$ for $n \geq 0$ correspond to the *discrete series representations* of $G_\mathbb{R}$, while the modules $D_{-1}^\pm$ correspond to the *limit of discrete series representations.*

We summarize the above considerations in the following theorem.

Theorem 9.1. *The irreducible Harish-Chandra modules for* $\mathrm{SL}_2(\mathbb{R})$ *are as follows:*

- *for each* $\lambda \in \mathbb{C}$ *with* $\operatorname{Re}\lambda \geq -1$ *and* $\lambda \neq -1$*, two modules* $P_\lambda^{\pm 1}$*,*
- *for each integer* $n \geq -1$*, two modules* $D_n^\pm$*.*