

Math 221 - Problem Set 6
Due Wednesday, Dec 4

All rings are commutative.

1. Characterize the p -adic expansions in \mathbb{Z}_p which are images of integers.
2. (a) Let R be a Noetherian ring. If R is local, and the maximal ideal \mathfrak{m} is principal, show that every nonzero ideal of R is a power of \mathfrak{m} .
(b) Deduce, without using the Principal Ideal Theorem, that if each maximal ideal of a Noetherian ring R is principal, $\dim R \leq 1$.
3. Let $R \rightarrow S$ be a homomorphism of rings, so that S is integral over R . Let M be an S -module. Show that the dimension of M as an S -module is equal to the dimension of M as an R -module.
4. Let R be a ring and $U \subset R$ multiplicatively closed. Define $S = R[U^{-1}]$. Show that for any prime $P \subset S$, the codimension of P in S is equal to the codimension of $P \cap R$ in R .
5. Let k be a field.
 - (a) Show that $\phi : k[x', y] \rightarrow k[x, y]$ defined $\phi(x') = x - y^n$ and $\phi(y) = y$ is an isomorphism of k -algebras.
 - (b) Let $f \in k[x, y]$. Show that if $n \gg 0$, then $\phi^{-1}(f) \in k[x', y]$ is a scalar times a monic polynomial in y over $k[x']$. [*Hint: the n you choose will depend on f . i.e. you can't choose f to be $x - y^n$.*]
 - (c) Deduce that $k[x, y]/(f)$ is integral over the subring $\phi(k[x'])$.
 - (d) Use (c) to prove that $\dim k[x, y] = 2$.
6. Let R be a Noetherian ring. Show that $\dim R[x, x^{-1}] = 1 + \dim R$.