Math 221 - Problem Set 5
Due Wednesday, Nov 20

All rings are commutative.

1. Let R be a ring and M an R-modules. Show that the following three statements are equivalent.

   i M is flat.

   ii $\operatorname{Tor}_i^R(M,N) = 0$ for all R-modules N and all $i > 0$.

   iii $\operatorname{Tor}_1^R(M,N) = 0$ for all R-modules N.

2. Let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. If $\{i,j,k\} = \{1,2,3\}$, for which values of i and j is the following statement true: If M_i and M_j are flat, then M_k is flat?

3. Calculate $\operatorname{Tor}_i^R(M,N)$ for all $i > 0$ in the following cases.

   (a) $R = \mathbb{C}[x,y]$, $M = R/(x)$, $N = R/(xy)$.

   (b) $R = \mathbb{Z}$, $M = \mathbb{Z}/(m)$, $N = \mathbb{Z}/(n)$.

   (c) $R = \mathbb{C}[x,y,z]$, $M = R/(x,y)$, $N = R/(x,y,z)$.

4. Let M and N be R-modules, and S a flat R-algebra. Show that

$$S \otimes \operatorname{Tor}_i^R(M,N) \cong \operatorname{Tor}_i^S(S \otimes M, S \otimes N).$$

   In particular, Tor commutes with localization.

5. Let R be Noetherian and M a finitely generated R module. Show M is flat if and only if $\operatorname{Tor}_1^R(M, R/P) = 0$ for every prime ideal P of R. (In fact, this is true for infinitely generated modules as well, by a reduction like the one we mentioned in class.)