Math 221 - Problem Set 4
Due Wednesday, Nov 6

All rings are commutative.

1. Let R be a ring and M an R-module. Let $\mathcal{J}$ be the filtration
$$M = M_0 \supset M_1 \supset \cdots .$$
   (a) Give an example to show that the function $\text{in} : M \to \text{gr}_{\mathcal{J}} M$ sending m to $\text{in}(m)$ is not necessarily a group homomorphism.
   (b) Let $f, g \in M$. Suppose for each i, f is in M_i if and only if $g \in M_i$. Show that either $\text{in}(f) + \text{in}(g) = \text{in}(f+g)$ or $\text{in}(f) + \text{in}(g) = 0$.
   (c) Suppose $M = R$ and that $\mathcal{J}$ is multiplicative, in the sense that $M_i M_j \subset M_{i+j}$. Show that either $\text{in}(f)\text{in}(g) = \text{in}(fg)$ or $\text{in}(f)\text{in}(g) = 0$.

2. Let $I \subset R$ be an ideal. Suppose $\text{gr}_I R$ is an integral domain. Let $J = (f) \subset R$.
   (a) Show that $\text{in}(J) \subset \text{gr}_I R$ is generated by $\text{in}(f)$.
   (b) Give an example of a local ring R with I maximal and $f \in I$ such that $\text{in}(J)$ is not generated by $\text{in}(f)$.

3. Suppose $J \subset I$ are ideals in a ring R. Show that
$$\text{gr}_I(R/J) = (\text{gr}_I R)/\text{in}(J).$$

4. Let R be a Noetherian ring and $I \subset R$ an ideal. Let M be a finitely generated R-module.
   (a) Show there is a largest submodule $N \subset M$ such that N is annihilated by an element of the form $1 - r$ with $r \in I$.
   (b) Show that $\bigcap_{j=1}^{\infty} I^j M = N$.