Math 221 - Problem Set 3
Due Wednesday, Oct 23

All rings are commutative.

1. Let R be a ring, S an R-algebra, M an S-module. Then M is naturally an R-module. If S is finite over R, and M is a finitely-generated S-module, show that M is a finitely-generated R-module. (We've been assuming this in class.)

2. Let k be a field and $R = k[t]/(t^2)$. Set
$$p(x) = tx^3 + tx^2 - x^2 - x \in R[x].$$
   (a) Show that $S = R[x]/(p)$ is a free R module of rank 2.
   (b) Recall a theorem that we proved in class (Prop 4.1 in the book) that says S is a free-module if and only if (p) is generated by a monic polynomial. Clearly p is not monic. How do you reconcile these two facts?

3. Let R be a local ring, with maximal ideal P. Let $I \subset R$ be an ideal, and suppose $x \in P$ is a nonzerodivisor on R/I.
   (a) Show that if I has a minimal set of generators, then it maps to a minimal generating set for the image of I in $R/(x)$.
   (b) Give an example to show that (a) can fail if x is a zerodivisor on R/I.

4. For each $n \in \mathbb{Z}$, find the integral closure R of $\mathbb{Z}[\sqrt{n}]$ as follows:
   (a) Show that you can reduce to the case where n is square-free.
   (b) Show that R is the integral closure of $\mathbb{Z}$ in the field $\mathbb{Q}[\sqrt{n}]$, and the minimal polynomial of $\alpha = a + b\sqrt{n}$ (with $a, b \in \mathbb{Q}$) is
$$m(x) = x^2 - (2a)x + (a^2 - b^2 n).$$
   Conclude $\alpha \in R$ if and only if the coefficients of m are integers.
   (c) Show that if $\alpha \in R$, then $a \in \frac{1}{2}\mathbb{Z}$. If $a = 0$, show $\alpha \in R$ if and only if $b \in \mathbb{Z}$. If $a = 1/2$ and $\alpha \in R$, show that $b \in \frac{1}{2}\mathbb{Z}$. Thus, subtracting a multiple of $\sqrt{n}$, we can assume $b = 0$ or $1/2$.
   (d) Conclude that $R = \mathbb{Z}[1/2 + 1/2\sqrt{n}]$ if $n \equiv 1 \pmod 4$, and $\mathbb{Z}[\sqrt{n}]$ otherwise.

5. Let R be an integral domain. Show R is normal if and only if $R[x]$ is normal.

6. Let $R = \mathbb{C}[x, y]/(y^2 - x^2(x+1))$, which corresponds to the nodal curve we saw in class. Set $t = y/x$.
   (a) Show that $R[t] = \mathbb{C}[t]$. Conclude that $R[t]$ is the normalization of R.
   (b) Consider the corresponding normalization map $\phi : \mathrm{Spec}(\mathbb{C}[t]) \to \mathrm{Spec}(R)$. Show that the point $(x, y) \in \mathrm{Spec}(R)$ (corresponding to the node at the origin) has exactly two points in its fiber. (In fact, it's an isomorphism away from that point!)