Math 221 - Problem Set 2
Due Wednesday, Oct 9

All rings are commutative.

1. Let R be a ring, $I \subset R$ an ideal, and $U \subset R$ a multiplicative set.

   (a) Show that the natural map $f : \mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ is a homeomorphism onto its image $V(I)$. That is, check that it is invertible, and that both f and f^{-1} send closed sets to closed sets.[1]

   (b) Show that the natural map $f : \mathrm{Spec}(R[U^{-1}]) \to \mathrm{Spec}(R)$ is a homeomorphism onto its image $Y = \{P \mid U \cap P = \emptyset\}$.

2. Let R be a ring, M an R-module. Define the ***support*** of M, $\mathrm{Supp}(M) \subset \mathrm{Spec}(R)$, to be the set of prime ideals P such that $M_P \neq 0$.

   (a) Show that if M is finitely generated, $\mathrm{Supp}(M)$ is the set of prime ideals containing the annihilator of M, and thus it is closed.

   (b) Let $R = k[x, y, z]$, k algebraically closed. Let $I = (x^2yz, z^2) \subset R$. Find the support and associated primes of R/I, and write I as the intersection of primary ideals.

3. Let R be a ring and M an R-module. Let $\{f_i\}$ be a set of elements of R that generate the unit ideal.

   (a) Show that if $m \in M$ goes to 0 in each $M[f_i^{-1}]$, then $m = 0$.

   (b) Show that if $m_i \in M[f_i^{-1}]$ are elements such that m_i and m_j go to the same element of $M[f_i^{-1}f_j^{-1}]$, then there is a unique element $m \in M$ such that m goes to m_i in $m_i \in M[f_i^{-1}]$ for each i.

4. Let k be a field. A ***monomial ideal*** is an ideal $I \subset k[x_0, x_1, \ldots, x_r]$ generated by monomials. Which monomial ideals are prime? Irreducible? Radical? Primary?

5. Here are some examples to show that prime avoidance can't be improved.

   (a) Show that if $k = \mathbb{Z}/(2)$, then the ideal $(x, y) \subset k[x, y]/(x, y)^2$ is the union of three properly smaller ideals.

   (b) Let k be any field. In the ring $k[x, y]/(xy, y^2)$, consider the ideals $I_1 = (x)$, $I_2 = (y)$, and $J = (x^2, y)$. Show that the homogeneous elements of J are contained in $I_1 \cup I_2$, but J is not contained in either.

6. Show that if k is an infinite field, the ideal (x, y) in $k[x, y]$ is contained in an infinite union of primes P_i such that no P_i contains (x, y). That is, prime avoidance can fail for an infinite number of primes.

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[1] The closed sets of $X \subset \mathrm{Spec}(R)$ are those of the form $V(J) \cap X$ for J an ideal in R.