Math 221 - Problem Set 1
Due Wednesday, Sept 25

All rings are commutative.

1. Prove that the following conditions on a module M over a ring R are equivalent.

   (i) M is Noetherian.

   (ii) Every ascending chain of submodules terminates.

   (iii) Every set of submodules of M contains elements that are maximal under inclusion.

   (iv) Given a sequence of elements $f_1, f_2, \ldots \in M$, there is some m such that for $n > m$, there is an expression
   $$f_n = \sum_{i=1}^{m} a_i f_i,$$
   with $a_i \in R$.

2. Let R be a Noetherian ring, and $I \subseteq R$ an ideal. A prime P containing I is ***minimal over*** I if there is no prime Q such that $I \subset Q \subsetneq P$. Show that there are only finitely many primes minimal over I . *[Hint: Assume not. Then there is an ideal I maximal among ideals for which it fails. This ideal can't be prime, so there are $f, g \in R - I$ such that $fg \in I$. Show that any prime maximal over I is maximal over (I, f) or (I, g).]*

3. Let k be a field. Compute the Hilbert function and polynomial for $k[x, y, z, w,]/(x, y) \cap (z, w)$.

4. Let R be a ring, $I \subset R$ an ideal, and $f \in R$. We define
   $$(I : f) := \{r \in R \mid fr \in I\}.$$

   (a) Check that $(I : f)$ is an ideal.

   (b) Let $U \subset R$ be a multiplicative set. Show that there is a one-to-one correspondence between the ideals in $R[U^{-1}]$ and ideals I in R such that $(I : f) = I$ for all $f \in U$.

   (c) Show that this correspondence sends prime ideals to prime ideals.

   (d) Show that for any ideal $J \subset R$, we have $R \cap JR[U^{-1}] = \sum_{f \in U}(J : f)$.

   (e) Show that the image of the map $J \mapsto R \cap JR[U^{-1}]$ is exactly the set of ideals $I \subset R$ such that $(I : f) = I$.

5. Let k be a field and m, n positive integers. Describe as explicitly as possible each of the following modules.

   (a) $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/(n), \mathbb{Z}/(m))$

   (b) $\mathrm{Hom}_{k[x]}(k[x]/(x^n), k[x]/(x^m))$

   (c) $\mathbb{Z}/(n) \otimes_{\mathbb{Z}} \mathbb{Z}/(m)$

(d) $k[x]/(x^n) \otimes_{k[x]} k[x]/(x^m)$

6. Show that the universal property of localization that we stated in class characterizes $R \to R[U^{-1}]$ up to unique isomorphism. That is if another map $R \to S$ satisfies the universal property, then there is a unique isomorphism $R[U^{-1}] \to S$ such that

$$\begin{array}{ccc} & R & \\ \swarrow & & \searrow \\ R[U^{-1}] & \longrightarrow & S \end{array}$$

commutes.