Math 137 - Problem Set 9
Due Friday, Apr 10

All rings are commutative, and k is an algebraically closed field.

1. Let P be a nonsingular (simple) point on a plane curve F. Show that the tangent line to F at P is
$$F_x(P)x + F_y(P)y + F_z(P)z = 0.$$
(Recall that the tangent line at a point is the projective closure of a tangent line in an affine chart containing that point.)

2. For each of the following projective plane curves, find their multiple points and the multiplicities and tangent lines at the multiple points.
   (a) $x^2y^3 + x^2z^3 + y^2z^3$
   (b) $y^2z - x(x-z)(x-\lambda z)$, $\lambda \in k$
   (c) $x^n + y^n + z^n$, $n > 0$

3. Let F be an irreducible projective plane curve. Show that F has only finitely many singular points.

4. Let $V = V(y - x^3)$ and $W = V(x)$ in $\mathbb{A}^2$.
   (a) Show that V and W are isomorphic as affine algebraic sets.
   (b) Show that the projective closures of V and W in $\mathbb{P}^2$ are not isomorphic. (Do you see what's going on geometrically?)

5. Find two projective plane curves in $\mathbb{P}^2$ that are isomorphic but have the property that when you dehomogenize with respect to z (i.e. intersect with U_3), the resulting affine plane curves are not isomorphic.

6. Let G be an irreducible cubic.
   (a) Show that G has at most one singular point, and any such singular point must have multiplicity 2.
   (b) Show that if G has a cusp (i.e., in this case, has a single tangent line of multiplicity 2 at the singular point), then it is projectively equivalent to the curve $y^2z - x^3$. (Hint: Using a problem from the previous problem set, you can start with the singular point at $[0:0:1]$ and the tangent line y.)
   (c) Show that if G has a node (i.e. has two distinct tangent lines at the singular point), then it is projectively equivalent to the curve $xyz = x^3 + y^3$.

7. **(Extra credit)** Let $H_1, H_2, \ldots, H_r$ be hyperplanes in $\mathbb{P}^n$. Show that the intersection $H_1 \cap \ldots \cap H_r$ is a linear space of dimension $\geq n - r$.