Math 137 - Problem Set 8
Due Friday, Apr 3

All rings are commutative, and k is an algebraically closed field.

1. Let $f \in k[x_1, \ldots, x_n]$ and $I = (f)$. Let $F \in k[x_1, \ldots, x_{n+1}]$ be the homogenization of f with respect to x_{n+1}, and let J be the homogenization of I (i.e. the ideal generated by the homogenization of elements of I). Show that $J = (F)$.

2. Let $V \subset \mathbb{A}^n$ be an affine algebraic set, and $I = I_a(V)$. Let $J \subset k[x_1, \ldots, x_{n+1}]$ be the homogenization of I. Define $\bar{V} \subset \mathbb{P}^n$ to be $V_p(J)$.

   (a) **(Extra credit)** Show that J is radical.

   (b) Show that $\bar{V} \cap U_{n+1} = V$ (where we've identified U_{n+1} and $\mathbb{A}^n$ in the natural way).

   (c) Show that $\bar{V}$ is equal to the Zariski closure of V in $\mathbb{P}^n$. That is, $\bar{V}$ is the smallest projective algebraic set containing V.

3. The ***twisted cubic*** $C \subset \mathbb{P}^3$ is the image of the map $\phi : \mathbb{P}^1 \to \mathbb{P}^3$ defined

$$\phi([s : t]) = [s^3 : s^2t : st^2 : t^3].$$

   (a) Show that C is the locus of points $[x : y : z : w]$ in $\mathbb{P}^3$ such that the rank of the matrix

$$\begin{pmatrix} x & y & z \\ y & z & w \end{pmatrix}$$

   is 1. (Varieties described as the vanishing of minors of a matrix are called ***determinantal varieties***.)

   (b) Show that no two of the minors in part (a) determine C. That is, all three are necessary.

4. Let $H = V(\sum a_i x_i)$ be a hyperplane in $\mathbb{P}^n$.

   (a) Show that assigning $[a_1 : \ldots : a_{n+1}] \in \mathbb{P}^n$ to H gives a one-to-one correspondence between hyperplanes in $\mathbb{P}^n$ and points in $\mathbb{P}^n$. This is called the ***dual projective space***. If $P \in \mathbb{P}^n$, let P^* be the corresponding hyperplane; if H is a hyperplane, let H^* be the corresponding point.

   (b) Show that $P \in H$ if and only if $H^* \in P^*$.

5. (a) Let $P_1, P_2, P_3 \in \mathbb{P}^2$ be three distinct points not lying on a line. Show that if $Q_1, Q_2, Q_3 \in \mathbb{P}^2$ are distinct points not lying on a line, there is a projective change of coordinates $T : \mathbb{P}^2 \to \mathbb{P}^2$ such that $T(P_i) = Q_i$. (Hint: Show there's a projective change of coordinates sending them to $[0 : 0 : 1]$, $[0 : 1 : 0]$, and $[1 : 0 : 0]$.)

   (b) State and prove an analogous result for two sets of three lines in $\mathbb{P}^2$. (What conditions do you need to impose on the three lines?)