Math 137 - Problem Set 6
Due Wednesday, Mar 11

All rings are commutative, and k is an algebraically closed field.

1. Let $k = \mathbb{C}$. Find the multiple points, and the tangent lines at the multiple points, for each of the following curves:

   (a) $y^3 - y^2 + x^3 - x^2 + 3xy^2 + 3x^2y + 2xy$

   (b) $x^4 + y^4 - x^2y^2$

   (c) $x^3 + y^3 - 3x^2 - 3y^2 + 3xy + 1$

2. The ***ring of formal power series over*** k is defined

$$k[[x]] := \left\{ \sum_{i=0}^{\infty} a_i x^i \mid a_i \in k \right\}.$$

   Show that $k[[x]]$ is a DVR with uniformizing parameter x.

3. If a curve f of degree n has a point P of multiplicity n, show that f consists of n lines through P (not necessarily distinct).

4. Let $T : \mathbb{A}^2 \to \mathbb{A}^2$ be a regular map, and $T(Q) = P$.

   (a) If $f \in k[x, y]$, show that the following inequality of multiplicities holds:

$$m_Q(T^*(f)) \geq m_P(f).$$

   (b) **(Extra credit)** Let $T = (T_1, T_2)$, and define

$$J_Q T = \begin{pmatrix} \frac{\partial T_1}{\partial x}(Q) & \frac{\partial T_1}{\partial y}(Q) \\ \frac{\partial T_2}{\partial x}(Q) & \frac{\partial T_2}{\partial y}(Q) \end{pmatrix}$$

   to be the ***Jacobian matrix*** of T at Q. Show that if $J_Q T$ is invertible, $m_Q(T^*(f)) = m_P(f)$. Does the converse hold?

You might want to wait until after class on Monday to work on the next two problems.

5. Let $P = (0, 0)$ and $k = \mathbb{C}$. Consider the following affine plane curves containing P:

   - $A = x^2 - y$
   - $B = y^2 - x^3 + x$
   - $C = y^2 - x^3$
   - $D = y^2 - x^3 - x^2$
   - $E = (x^2 + y^2)^3 - 4x^2y^2$

(a) Compute $I_P(A, C)$.
(b) Compute $I_P(C, D)$.
(c) Compute $I_P(B, E)$.

6. Let f, g, and h be affine plane curves.

   (a) If P is a simple point on f, show

   $$I_P(f, g + h) \geq \min\{I_P(f, g), I_P(f, h)\}.$$

   (b) Give an example to show that (a) may be false if P is not simple on f.

Bonus questions (Extra credit)

7. Let $P = (0, 0)$ lie on an irreducible curve f. Let $\mathfrak{m} = \mathfrak{m}_P(f)$ be the maximal ideal of $\mathcal{O}_P(f)$.

   (a) Show that $\dim_k(\mathfrak{m}^n/\mathfrak{m}^{n+1}) = n + 1$ for $0 \leq n < m_P(f)$.
   (b) Conclude that P is a simple point if and only if $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 1$; otherwise $\dim_k(\mathfrak{m}/\mathfrak{m}^2) = 2$.

8. Let X and Y be affine varieties, and $\phi : X \to Y$ a regular map. We showed on the last homework that X and Y are locally ringed spaces, with structure sheaves $\mathcal{O}_X$ and $\mathcal{O}_Y$, respectively. Look up the definition of a morphism between locally ringed spaces (e.g. here), and show that ϕ is a morphism of locally ringed spaces.