Math 137 - Problem Set 5
Due Wednesday, Mar 4

All rings are commutative, and k is an algebraically closed field.

1. Let $X \subset \mathbb{A}^n$ be a nonempty linear subvariety (i.e. its corresponding ideal is generated by nonzero polynomials of degree 1).

   (a) Show that there is some change of coordinates $T : \mathbb{A}^n \to \mathbb{A}^n$ such that $T(X) = V(x_{m+1}, \ldots, x_n)$, for some $m < n$.

   (b) Conclude that X is isomorphic to $\mathbb{A}^m$.

2. Let $\mathcal{O}_P(V)$ be the local ring of a variety V at a point P. Show that there is a natural one-to-one correspondence between the prime ideals in $\mathcal{O}_P(V)$ and the subvarieties of V that pass through P.

3. Let $T : \mathbb{A}^n \to \mathbb{A}^n$ be an affine change of coordinates. Let V be a subvariety of $\mathbb{A}^n$, and $P \in V$ a point and $Q = T(P)$.

   (a) Show that the map $\mathcal{O}_Q(\mathbb{A}^n) \to \mathcal{O}_P(\mathbb{A}^n)$ induced by T^* is an isomorphism[1]. [Hint: Use the fact that T is an isomorphism. Your solution should be pretty short.]

   (b) Show that $T(V)$ is a closed subvariety of $\mathbb{A}^n$.

   (c) Show that the map $\mathcal{O}_Q(T(V)) \to \mathcal{O}_P(V)$ induced by T^* is an isomorphism.

4. Let f be a rational function on an affine variety V. Let

   $$U = \{P \in V \mid f \text{ is defined at } P\}.$$

   Recall that U is open, since we showed in class that its complement is closed. Then f defines a function from U to k. Show that this function determines f uniquely.

5. Let $X \subset \mathbb{A}^n$ be a variety. You may (but don't need to) assume $\Gamma(X)$ is a UFD. Let $U \subset X$ be a Zariski open set in X (i.e. the complement of some algebraic subset of X). Define $\mathcal{O}_X(U) \subset k(X)$ to be the set of rational functions that are regular (i.e. defined) at every point of U.

   (a) Show that $\mathcal{O}_X(U)$ is a ring (In fact, a k-algebra).

   (b) For $f \in \Gamma(X)$, define the open set

   $$U_f := X - V(f).$$

   Describe explicitly the ring $\mathcal{O}_X(U_f)$.

   (c) If $V \subset U$ is an open set contained in U, describe the natural ring homomorphism

   $$\phi_{U,V} : \mathcal{O}_X(U) \to \mathcal{O}_X(V).$$

[1] We described this map in class.

6. Let $X = \mathbb{A}^2$.

   (a) Describe the ring $\mathcal{O}_X(U)$, where $U \subset \mathbb{A}^2$ is the complement of the x-axis.

   (b) Describe the ring $\mathcal{O}_X(V)$, where V is the complement of the origin. (Do you see why your answer makes sense?)

Bonus questions (Extra credit)

7. Look up the definition of a sheaf. Show that $\mathcal{O}_X$ described in problem 5 is a sheaf of rings[2] (in fact, of k-algebras) on X. (You've already done a lot of the work above.)

8. Look up the definition of the stalk of a sheaf. Show that the stalk of $\mathcal{O}_X$ at a point $P \in X$ is the local ring[3] $\mathcal{O}_P(X)$. This gives X the structure of a ***locally ringed space***.

[2] The sheaf $\mathcal{O}_X$ is called the ***structure sheaf*** of X.

[3] The local ring is often denoted $\mathcal{O}_{P,X}$ to avoid notation confusion.