Math 137 - Problem Set 3
Due Wednesday, Feb 19

All rings are commutative, and k is an algebraically closed field.

1. Let $R \subset S \subset T$ be integral domains.

   (a) If S is module-finite over R and T is module-finite over S, prove that T is module-finite over R.

   (b) If S is integral over R and T is integral over S, prove that T is integral over R.

2. Let $X = V(f)$ be an irreducible hypersurface in $\mathbb{A}^n$. Show that if $Y \subset \mathbb{A}^n$ is an affine variety containing X, then $Y = \mathbb{A}^n$ or $Y = X$.

3. Let $R = \mathbb{C}[x, y]$. For each ideal $I \subset R$, find $V(I)$, and compute $\dim_k(R/I)$.

   (a) $I = (y^2 - x^3, y - x^2)$

   (b) $I = (y^2 - x^2, y^2 + x^2)$.

4. Let $V \subset \mathbb{A}^n$ be a nonempty variety. Recall that the ***ring of regular functions*** on V, denoted $\Gamma(V)$, is the set of functions $f : V \to k$ such that $f = F|_V$, where $F \in k[x_1, \ldots, x_n]$, and $F|_V$ denotes the restriction of F to V. Let $\phi : k[x_1, \ldots, x_n] \to \Gamma(V)$ be the restriction map $F \mapsto F|_V$.

   (a) Check that ϕ is a k-algebra homomorphism (i.e. a ring homomorphism that acts as the identity on k).

   (b) Show that $\ker(\phi) = I(V)$.

   (c) Conclude that $\Gamma(V) \cong k[x_1, \ldots, x_n]/I(V)$.

5. Let $V \subset \mathbb{A}^n$ be a nonempty variety. Show that the following are equivalent.

   (i) V is a single point;

   (ii) $\dim_k \Gamma(V) < \infty$;

   (iii) $\Gamma(V) = k$.