Math 137 - Problem Set 2
Due Wednesday, Feb 12

All rings are commutative, and k is an algebraically closed field.

1. Let I be an ideal in a ring R. Show that there is a one-to-one correspondence between radical ideals in R containing I and radical ideals in R/I. (Convince yourself that the same holds for prime ideals. You don't need to write up that part.)

2. (a) Let $I \subset k[x_1, \dots, x_n]$ be an ideal. Show that I is radical if and only if it is equal to the intersection of all the maximal ideals containing it.

   (b) Show that the radical of the ideal $I = (x^2 - 2xy^4 + y^6, y^3 - y) \subset \mathbb{C}[x, y]$ is the intersection of three maximal ideals.

3. Let $X = V(x^2 - yz, xz - x) \subset \mathbb{A}^3_{\mathbb{C}}$. Find the irreducible components of X and their corresponding prime ideals. Make sure you justify your solution.

4. Let $a_1, a_2, \dots a_n \in k$. Show that $(x_1 - a_1, \dots, x_n - a_n) \subset k[x_1, \dots, x_n]$ is a maximal ideal. (Hint: reduce to the case where the a_i are all 0.)

5. Let $X \subset \mathbb{A}^n$ be a set (not necessarily algebraic). The ***Zariski closure*** of X, denoted $\overline{X}$, is the intersection of all Zariski closed sets containing X. Show that $V(I(X)) = \overline{X}$.

6. A subset of affine space $U \subset \mathbb{A}^n$ is called ***compact*** (in the Zariski topology) if for every collection $\{U_i\}_{i \in J}$ (where J is some indexing set) of Zariski open sets such that if
$$U \subset \bigcup_{i \in J} U_i,$$
then U is also contained in some finite union of the U_i. That is, there is some finite set $L \subset J$ such that
$$U \subset \bigcup_{i \in L} U_i.$$
More concisely, U is compact if every open cover has a finite subcover. Show that if $X \subset \mathbb{A}^n_k$ is an algebraic set, X is compact in the Zariski topology.

Bonus topology questions (Extra credit)

7. Identify $\mathbb{A}^1 \times \mathbb{A}^1$ with $\mathbb{A}^2$ in the natural way. Show that the Zariski topology on $\mathbb{A}^2$ is not the product topology induced by the Zariski topology on $\mathbb{A}^1$.

8. For each $f \in k[x_1, \dots, x_n]$ define U_f to be the set of points $P \in \mathbb{A}^n$ such that $f(P) \neq 0$. Prove that the collection of all such sets U_f forms a basis for the Zariski topology on $\mathbb{A}^n$.