Math 137 - Problem Set 11
Due Wednesday, Apr 29

All rings are commutative, and k is an algebraically closed field.

1. Show that $V = \mathbb{A}^n - \{0\}$ is an affine variety if and only if $n = 1$. [1]

2. Let C be the affine plane curve $V(x^3 + x^2 - y^2) \subset \mathbb{A}^2$.

   (a) Show that C is rational by explicitly describing a birational map $f : \mathbb{A}^1 \dashrightarrow C$ and checking that it induces an isomorphism of fields $f^* : k(C) \to k(\mathbb{A}^1) = k(t)$

   (b) Which DVR(s) in $k(t)$ dominate the local ring $f^*(\mathcal{O}_{(0,0)}(C))$?

3. Let X and Y be varieties, and let $f : X \to Y$ be a function. Suppose $X = \cup_\alpha U_\alpha$ and $Y = \cup_\alpha V_\alpha$, where $U_\alpha \subset X$ and $V_\alpha \subset Y$ are open, and $f(U_\alpha) \subset V_\alpha$ for each α.

   (a) Show that f is a morphism if and only if each restriction $f_\alpha : U_\alpha \to V_\alpha$ of f is a morphism. [2]

   (b) If each U_α and V_α is an affine variety, show that f is a morphism if and only if each $f^*(\Gamma(V_\alpha)) \subset \Gamma(U_\alpha)$.

4. Define a morphism $f : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$ by $f([x_0 : x_1], [y_0 : y_1]) = [x_0y_0 : x_0y_1 : x_1y_0 : x_1y_1]$. This map is called the ***Segre embedding*** of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^3$.

   (a) Check that this map is well-defined and injective.

   (b) **(Extra credit)** Show that f is a morphism.

   (c) Let the homogeneous coordinates of $\mathbb{P}^3$ be t_{00}, t_{01}, t_{10}, and t_{11}. Define the ideal

   $$I = (t_{01}t_{10} - t_{00}t_{11}) \subset k[t_{00}, t_{01}, t_{10}, t_{11}].$$

   Show that the image of f is $V(I)$.

   (d) **(Extra credit)** Generalize the above to $\mathbb{P}^n \times \mathbb{P}^m$. What is the ideal of the image of the map in this case?

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[1] Hint: what is $\Gamma(V)$?

[2] Your answer here should not be super technical and complicated. The crux of this problem is just parsing definitions.