Math 137 - Problem Set 10
Due Friday, Apr 17

All rings are commutative, and k is an algebraically closed field.

1. (a) Let $P_1, \dots, P_r \in \mathbb{P}^2$ and $d \geq 1$. Show that there is a plane curve of degree d that doesn't contain any of the P_i.
   (b) Does (a) generalize to hypersurfaces of degree d in $\mathbb{P}^n$?

2. (a) Let Y be a set of 5 distinct points in $\mathbb{P}^2$. Let V be the linear system of conics that contain Y. Show that $\dim(V) > 0$ if and only if at least four of the points are collinear.[1]
   (b) Let Z be a set of 10 distinct points in $\mathbb{P}^2$. Let W be the (possibly empty!) linear system of cubics that contain Z. Show that $\dim(W) > 0$ if and only if at least 6 of the points are collinear or at least 9 of the points lie on a conic.

3. Let F be an irreducible plane curve of degree d. Assume the partial derivative $F_x \neq 0$.
   (a) If P is a point on F, show that $m_P(F_x) \geq m_P(F) - 1$.
   (b) Using part (a) along with Bézout's theorem, show that
   $$\sum_{P \in V(F)} m_P(m_P - 1) \leq d(d-1).$$
   (c) Conclude that F has at most $\frac{1}{2}d(d-1)$ multiple points.
   (d) Give an example to show that the bound in (c) is not sharp. That is, show that there is some d such that F cannot have $\frac{1}{2}d(d-1)$ multiple points.

4. Let $n \geq 1$ and $d \geq 1$. Define a map $v_d : \mathbb{P}^n \to \mathbb{P}^N$ by
   $$v_d([x_0 : x_1 : \dots : x_n]) = [M_0 : M_1 : \dots : M_N],$$
   where the M_i are all the degree d monomials in $n+1$ variables. That is,
   $$M_0 = x_0^d,\ M_1 = x_0^{d-1}x_1,\ \dots,\ M_N = x_n^d.$$
   This map is called the ***degree d Veronese embedding*** of $\mathbb{P}^n$.
   (a) What is N? Your answer should be in terms of d and n.
   (b) Show that v_d is well-defined and injective.[2]
   (c) Denote $V := v_d(\mathbb{P}^n)$, the image of $\mathbb{P}(n)$. You may assume V is Zariski closed. Let $H \subset \mathbb{P}^N$ be a hyperplane. Show that $H \cap V = v_d(W)$, where $W \subset \mathbb{P}^n$ is a hypersurface of degree d.[3]

[1] Hint: Remember that $\dim(V) = 0$ if and only if there is *exactly* one conic through those points.
[2] In fact, it's an isomorphism onto its image. You don't need to prove this.
[3] $H \cap V$ is called a ***hyperplane section*** of V.

(d) Show that the converse of (c) holds. Specifically, let $W \subset \mathbb{P}^n$ be a hypersurface of degree d. Show that there is some hyperplane $H \subset \mathbb{P}^N$ such that $H \cap V = v_d(W)$. That is, the hyperplane sections of V are exactly the degree d hypersurfaces in $\mathbb{P}^n$.

(e) Notice that the twisted cubic in $\mathbb{P}^3$ is the degree 3 Veronese embedding of $\mathbb{P}^1$.[4] Apply the last sentence in (d) to the case of the twisted cubic. What does it say, concretely, in this case?

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[4]The Veronese embeddings of $\mathbb{P}^1$ are called ***rational normal curves***.