**Connected spaces**

**Def:** A topological space is connected if it cannot be written \(X = U \cup V\) where \(U\) and \(V\) are disjoint nonempty open sets (called a separation of \(X\)).

**Ex:** \([0,1]\) is connected (w/ standard topology).

Suppose \([0,1] = U \cup V\). WLOG \(0 \in U\).

Let \(a = \sup \{x \in [0,1] \mid (0, x) \subseteq U\}\).

\(U\) is open, so \(a > 0\). If \(a = 1\), we're done.

Otherwise, \(a \in U\). But then a neighborhood of \(a\) is in \(V\) so we can't have \([0, a) \subseteq U\).

**Ex:** \([0,1] \cup (1,2]\) is not connected, since both \([0,1]\) and \((1,2]\) are open (in the subspace topology).

**Ex:** Consider \(\mathbb{R}_e\). \(\mathbb{R}_e = (-\infty, 0) \cup [0, \infty)\), both open, so \(\mathbb{R}_e\) is not connected.

In fact, every subspace of \(\mathbb{R}_e\) is disconnected other than single points, i.e., \(\mathbb{R}_e\) is totally disconnected.
Note that connectedness is not preserved in subspaces:

\[ X = \{ y = x^2 \} \quad \text{is connected in } \mathbb{R}^2 \]

but \( X \cup Y \) is not connected in \( Y \).

However, it’s preserved by continuous functions:

**Thm:** If \( f : X \to Y \) is continuous and \( X \) connected, then \( f(X) \) is connected.

**Pf:** Since the map \( X \to f(X) \) is continuous as well, we can assume \( X \) is surjective.

Suppose \( Y = U \cup V \) is a separation of \( Y \). Then \( f^{-1}(U) \) and \( f^{-1}(V) \) are open, nonempty, disjoint, and their union is \( X \). \( \square \)

Certain unions of connected spaces are also connected:

**Thm:** If \( A_i \subseteq X \) are connected subspaces that all have a point in common, then \( Y = \bigcup A_i \) is connected.

**Pf:** Suppose \( Y = U \cup V, U \) and \( V \) both open.

Suppose the common point \( p \) is in \( U \).

\( U \cap A_i \) and \( V \cap A_i \) are disjoint open sets in \( A_i \).

Since \( A_i \) is connected and \( p \in A_i \), \( A_i \subseteq U \forall i \)

\[ \Rightarrow \quad Y = \bigcup A_i \subseteq U \Rightarrow \quad Y \text{ is connected}. \] \( \square \)
Cor: \( \mathbb{R} \) is connected, as are all open, half open, and closed intervals in \( \mathbb{R} \).

Pf: \([0, 1]\) is connected and homeomorphic to all \([a, b]\). 
\[ \bigcup_{n \in \mathbb{N}} [-n, n] = \mathbb{R} \text{ is connected, since } [-n, n] \text{ all contain 0}. \]

Recall the intermediate value Thm from calculus:
If \( f: [a, b] \rightarrow \mathbb{R} \) is continuous, then \( \forall y \) between
\( f(a) \) and \( f(b) \), \( \exists c \in [a, b] \) s.t. \( f(c) = y \).

Key point: Connectedness of \([a, b]\) requires the image to be connected.

More generally...

Thm: (Intermediate value theorem) Let \( X \) be a connected topological space, and \( f: X \rightarrow \mathbb{R} \) continuous.
If \( a, b \in X \) and \( r \) lies between \( f(a) \) and \( f(b) \) in \( \mathbb{R} \),
Then \( \exists c \in X \text{ s.t. } f(c) = r. \)

\[ X \quad \text{\textbullet} \quad c \quad \text{\textbullet} \quad b \quad \text{\downarrow} \quad f \quad \text{\textbullet} \quad f(a) \quad \text{\textbullet} \quad f(c) \quad \text{\textbullet} \quad f(b) \quad \text{\IR} \]

**Pf:** Since \( X \) is connected, so is \( f(X) \).

Consider \( U = (-\infty, r) \cap f(X) \) and \( V = (r, \infty) \cap f(X) \).

Both are open in \( f(X) \) and nonempty, since \( f(a) \) is in one, \( f(b) \) in the other.

If \( r \notin f(X) \), then \( f(X) = U \cup V \), a contradiction. Thus, \( r \in f(X) \).

So \( \exists c \in X \text{ s.t. } f(c) = r. \quad \Box \)

**Products of connected spaces**

**Thm:** \( X, Y \) connected \( \Rightarrow X \times Y \) connected.

**Pf:** Suppose \( X \times Y = U \cup V \), \( F(x) (a, b) \in U. \)

Let \( (a', b') \in X \times Y. \)

Then \( X \times \{b'\} \text{ is } \IR \).
connected, so $(a', b) \in U$.

And $\{a' \times Y \}$ is connected, so $(a', b') \in U$.

Since $(a', b')$ was arbitrarily chosen, $X \times Y \in U \Rightarrow X \times Y$ is connected.

**Cor:** Finite products of connected spaces are connected.

**Pf.**

$X_1 \times X_2 \times \ldots \times X_n = (X_1 \times X_2 \times \ldots \times X_{n-1}) \times X_n$. □

What about infinite products?

**Claim:** If $\{X_i\}_{i \in I}$ is a collection of connected spaces, then $\prod_{i \in I} X_i$ is connected given the product topology.

(Exercise)

This is not true in the box topology...

**Ex.** Consider $X = \mathbb{R}^\omega$ given box topology.

Let $U$ be the set of bounded sequences, i.e. $(a_1, a_2, \ldots)$ s.t. $\exists N$ s.t. $|a_i| \leq N \ \forall \ i$. 
$V$ the set of unbounded sequences.

Clearly $UUV = X$.

Why is $U$ open?

Let $\hat{a} = (a_1, a_2, \ldots) \in X$. Consider

$Y = (a_1 - 1, a_1 + 1) \times (a_2 - 1, a_2 + 2) \times \ldots$

$Y$ is open, and if $\hat{a} \in U$, $Y \subseteq U$. If $\hat{a} \in V$, $Y \subseteq V$.

Thus, $U$ and $V$ are both open, so the box topology is not connected.

Path connected spaces

**Def.** If $X$ is a topological space and $x, y \in X$, a path from $x$ to $y$ is a continuous map $f: [a, b] \to X$ s.t. $f(a) = x$ and $f(b) = y$.

$X$ is path connected if every pair of points in $X$ can be joined by a path.

**Note:** The relation $x \sim y \iff x$ and $y$ can be connected by
a path is an equivalence relation:
1.) $x \sim x$ by the constant path $f(t) = x$
2.) $x \sim y \iff y \sim x$ since we can run the path backward
3.) $x \sim y$, $y \sim z \Rightarrow x \sim z$ by running one path and then the next.

The equivalence classes are called path components. We'll use these a lot when we get to algebraic topology.

**Ex:**

![Diagram]

- path connected
- not path connected

How is path connectedness related to connectedness?

**Thm:** If $X$ is path connected it's connected.

**Pf:** Suppose $X = U \cup V$, $x \in U$. Pick a path $f : [a, b] \to X$ connecting $x$ to some other point $y$. Then $f([a, b])$ is connected, so $f([a, b]) \subseteq U$. Thus $y \in U \cap y \in X$. $\square$

The converse does not hold in general!!

**Ex:** Let $S \subseteq \mathbb{R}^2$ be defined:

$$S = \left\{ (x, y) \mid y = \sin(\pi x) \right\} \cup (0, 0)$$
This is called the topologist's sine curve.

A is connected, as it's the image of a connected space. Any point \((0,0)\) is a limit point of \(A\), since \(A\) is connected.

There is some \(N\) s.t. \(\sin\left(\frac{1}{N}\right) = 0\) and \(N < \varepsilon\).

Since \(A\) is connected, and \((0,0)\) is a limit point of \(A\), \(S\) is connected. Otherwise, if \(S = U \cup V\), \(A \subseteq U\), and if \((0,0) \in V\) then \(V \cap A \neq \emptyset\). Thus, \(S = U\) and \(S\) is connected.

However, \(S\) is not path connected! Consider \(x = (0,0)\), \(y \in A\). There is no path connecting \(x\) to \(y\). (Exer)

Idea: Find \(x_1, x_2, \ldots \in [a, b]\) s.t.

\[
x_1, x_2, \ldots \to a, \quad \text{but} \quad f(x_1) = f(x_2) = \ldots = 1 \to 1 \neq 0.
\]
For "well-behaved" spaces, connectedness is the same as path connectedness. e.g. manifolds.

**Thm:** If \( U \subseteq \mathbb{R}^n \) is open, then \( U \) is connected \( \iff \) path connected.

**Pf:** We already know "\( \subseteq \)."

"\( \Rightarrow \)" : Suppose \( U \) is connected, \( x \in U \).

Let \( V \subseteq U \) be the set of points that can be reached from \( x \) by a polygonal path (i.e. a union of line segments).

\( V \) is open since for \( z \in V \), any point in a ball \( B_e(z) \subseteq U \) can be reached by a straight line from \( z \).

Claim: \( V \) is also closed in \( U \). If \( y \in \overline{V} \), then there is a ball \( y \in B \subseteq U \), s.t. \( B \cap V \neq \emptyset \). so there is a \( p \in B \cap V \) that can be reached by \( y \) and by \( x \). \( \Rightarrow y \in V \Rightarrow V \) is closed and open \( \Rightarrow V = U. \) \( \square \)