The Quotient Topology

We can get lots of interesting examples of topological spaces by "gluing" together simple spaces we already know.

Ex:

[Diagrams of simple spaces being glued together]

We need to formalize this construction.

Def: Let $X$ be a topological space and $A$ a set.

Let $f: X \to A$ be a surjective function. The quotient topology on $A$ is defined by $U \subseteq A$ is open $\iff f^{-1}(U)$ is open. (Exer: check this is a topology)

A map $f: X \to Y$ between topological spaces is a quotient map if $f$ is surjective and $f^{-1}(U)$ is open $\iff U \subseteq Y$ is open.

Note that with the quotient topology on $A$, $f: X \to A$ is a quotient map. $A$ is called a quotient space of $X$.

We can also construct $A$ from $X$ by introducing an equivalence relation $\sim$ on $X$ and setting $A = X/\sim$. Then $f: X \to A$ sends $x$ to the equivalence class containing $x$. 
**Ex:** We can think of $S^1$ as $[0,1]$ with 0 glued to 1. i.e. the equivalence relation is just $0 \sim 1$, and the quotient map is $f(x) = (\cos 2\pi x, \sin 2\pi x)$.

Note that the map $g: [0,1) \to S^1$ defined as the restriction of $f$ to $[0,1)$ is also surjective, but is not a quotient map:

Let $U = \{(x,y) \mid y > 0\} \cup \{(0,0)\} \subseteq S^1$.

Then $U$ is not open, but $g^{-1}(U) = [0, \frac{1}{2}) \subseteq [0,1)$ is open! (whereas $f^{-1}(U) = [0, \frac{1}{2}) \cup \{1\}$ is not).

**Ex:** Let $(X_1, x_1), \ldots, (X_n, x_n)$ be pointed topological spaces w/ $X_i$ homeomorphic to $S^1$.

Then we get a quotient space $A$ of $\sqcup X_i$ by the equivalence relation $x_i \sim x_j \forall i, j$. This is called the wedge of the circles $X_1, \ldots, X_n$.

A nice property of quotient maps is that if $h: X \to Y$ is a map that "respects the quotient structure" $X \to A$, we can uniquely define a map $A \to Y$. More precisely...
**Thm:** let $p : X \to Y$ be a quotient map. let $f : X \to \mathbb{Z}$ be a continuous map to a topological space $\mathbb{Z}$ such that $f$ is constant on each set $p^{-1}(\{y\})$ for $y$ in $\mathbb{Z}$. (i.e. if $p(x) = p(x')$, then $f(x) = f(x')$). Then there is a continuous map $g : Y \to \mathbb{Z}$ s.t. $g \circ p = f$. i.e. 

\[
\begin{array}{c}
Y \\
\downarrow g \\
\mathbb{Z}
\end{array}
\xrightarrow{\text{commutes}}
\begin{array}{c}
X \\
\downarrow p \\
Y
\end{array}
\]

**Pf:** For each $y \in Y$, $f(p^{-1}(\{y\}))$ is a one-point set. Define $g(y)$ to be this point. Then if $x \in X$, $g(p(x)) = f(x)$ by construction. Thus, $g \circ p = f$.

For continuity, let $U \in \mathbb{Z}$ be an open set. We want to show $g^{-1}(U)$ is open. $g^{-1}(U) \subseteq Y$ is open $\iff$ $p^{-1}(g^{-1}(U)) \subseteq X$ is open. But $p^{-1}(g^{-1}(U)) = (g \circ p)^{-1}(U) = f^{-1}(U)$, which is open since $f$ is continuous. Thus $g^{-1}(U) \subseteq Y$ is open.

**Note:** If $f : X \to Y$ is surjective, continuous and open, then $f$ is a quotient map. Similarly, if $f$ is closed, then if $f^{-1}(U)$ is open, $X \setminus f^{-1}(U) = f^{-1}(Y \setminus U)$ is closed, so $Y \setminus U$ is closed $\Rightarrow U$ is open.

Thus $f$ is also a quotient map in this case.

**Example:** let $X = \mathbb{R}^n \setminus \{0\}$. We can put an equivalence relation on $X$ as follows: $x \sim y \iff x = ay$ i.e. $x \sim y$ if and only if $x$ and $y$ lie on the same line through the origin.
You can check that this is an equivalence relation, and thus we get a corresponding quotient map 
\[ p: X \to X/\sim. \]

This is one way to construct projective \((n-1)\)-space, defined
\[ \mathbb{RP}^{n-1} := X/\sim, \]
and the quotient topology.

If \( Z \) is another topological space, then the continuous maps 
\[ f: X \to Z \]
that give an induced map \( \mathbb{RP}^{n-1} \to Z \)
have the property that \( f(\alpha x) = f(x) \) \( \forall \alpha \in \mathbb{R}\setminus\{0\}, x \in X. \)

On the HW, we'll see that \( \mathbb{RP}^1 \) is homeomorphic to \( S^1 \).
But we'll later see that \( \mathbb{RP}^n \) is not homeomorphic to \( S^n \) for \( n > 1 \).

**Example:** Quotients of the unit square \( X = [0,1]^2 \)

\[
\begin{align*}
A &= \{0,3\} \times [0,1] \\
A' &= \{1,3\} \times [0,1] \\
B &= \{0,1\} \times \{0,3\} \\
B' &= \{0,1\} \times \{1,3\}
\end{align*}
\]

1) If we glue \( A \) to \( A' \) via the equivalence relation \((0,b) \sim (1,b)\),
we get a cylinder. A typical neighborhood of a point on the gluing line corresponds to two half moons along \( A \) and \( A' \) in \( X \).
2.) If we glue $A$ to $A'$ via $(0,t) \sim (1,1-t)$, we get a **Möbius band**.

3.) As we've seen, gluing $A$ to $A'$ and $B$ to $B'$ via $(0,t) \sim (1,t)$ and $(s,0) \sim (s,1)$ gives us the torus.

4.) Gluing via $(0,t) \sim (1,t)$ and $(s,0) \sim (1-s,1)$ gives us the Klein Bottle, which cannot be embedded in $\mathbb{R}^3$! We draw a picture where it overlaps itself, whereas an actual Klein bottle does not.

5.) Gluing $(0,t) \sim (1,1-t)$ and $(s,0) \sim (1-s,1)$ is a lot trickier to visualize! It turns out this space is homeomorphic to $\mathbb{RP}^2$!!

**Exercise:** Consider the quotient spaces $[0,1] \times [0,1]/\sim$ with the following equivalence relations.

a.) $(0,t) \sim (t,0)$

b.) $(0,t) \sim (t,0)$ and $(1,s) \sim (s,1)$

c.) (HARD) $(t,0) \sim (0,1-t)$
Can you visualize these spaces?