Math 131 - Problem Set 8
Due Thursday, Nov 8

From Munkres: 54.8, 22.2

1. For X a topological space and $A \subset X$, let X/A denote the quotient space of X where all points in $A \subset X$ are identified to a single point. Prove that $\mathbb{R}^2/A$ is homeomorphic to $\mathbb{R}^2$ when $A = B^2 (= \{x \mid \|x\| \leq 1\})$, but not when $A = S^1$.

2. Let $X = \mathbb{R}^{n+1} - \{0\}$, $n \geq 0$. In class, we defined an equivalence relation on X by $x \sim y \iff x = \alpha y$ for some $\alpha \in \mathbb{R}$. The quotient space $X/\sim$ is called projective n-space, denoted $\mathbb{RP}^n$.

   (a) Show that $\mathbb{RP}^n$ is homeomorphic to the quotient space $S^n/\sim$ where $a \sim b \iff a = -b$.

   (b) Show that $\mathbb{RP}^1$ is homeomorphic to S^1.

3. Let E, F, and B be topological spaces. A **fiber bundle** with fiber F is a continuous map $f : E \to B$, where every $b \in B$ has an open neighborhood $U \subset B$ satisfying the following:

   - There is a homeomorphism $h : f^{-1}(U) \to U \times F$.
   - If $\pi : U \times F \to U$ is the projection onto the first factor, $f|_{f^{-1}(U)} = \pi \circ h$.

   (a) Show that a fiber bundle $f : E \to B$ is a quotient map.

   (b) Show that the Möbius band (i.e. the quotient of $[0, 1] \times [0, 1]$ by the relation $(0, y) \sim (1, 1 - y)$) is a fiber bundle over S^1.

   (c) Show that if B is connected, $f : E \to B$ is a covering map if and only if f is a fiber bundle with fiber F, where F has the discrete topology.