Math 131 - Problem Set 1
Due Tuesday, Sept 11

1. Show that an infinite sequence of points $a_1, a_2, a_3, \ldots$ in a metric space has at most one limit point.

2. (a) Show that any sequence in a metric space with a limit is a Cauchy sequence.

   (b) Show (with an example) that the converse of (a) is not true.

3. Recall that a subset U of a metric space X is open iff for all $x \in U$, there is some $\epsilon > 0$ such that $x \in B_\epsilon(x) \subset U$. Show that any open ball is in fact an open set.

4. Let X be a metric space. Show that the collection of open sets on X (given by the definition in the previous exercise) is a topology.

5. Consider the following distance functions on $\mathbb{R}^n$:

$$e((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \max\{|y_i - x_i|\}$$

   and

$$f((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n)) = \sum |y_i - x_i|.$$

   (a) Show that $(\mathbb{R}^n, e)$ and $(\mathbb{R}^n, f)$ are metric spaces.

   (b) Show that e and f induce the same topology on $\mathbb{R}^n$ (i.e. a set is open in one iff it's open in the other).

6. Let X be a metric space. Show that a sequence $p_1, p_2, p_3, \ldots$ of points in X has limit p if and only if every open set containing p contains all but finitely many of the p_i. (This gives a characterization of limits using open sets without explicitly using the metric.)

7. Let X be a topological space, with basis $\mathcal{B}$ generating the topology $\mathcal{T}$. Show that $\mathcal{T}$ is the smallest topology on X that contains $\mathcal{B}$. That is, every other topology containing $\mathcal{B}$ is finer than $\mathcal{T}$. (*We'll define bases in class on Thursday, Sept 6.*)

8. Give an example of a topological space X and an infinite sequence of bases

$$\mathcal{B}_1 \supsetneq \mathcal{B}_2 \supsetneq \mathcal{B}_3 \supsetneq \cdots$$

   all generating the same topology $\mathcal{T}$.