Math 122 - Problem Set 9
Due Wednesday, Nov 13

1. Which of the following are subrings of $\mathbb{Q}$? Be sure to justify your answer. An answer with no justification won't receive any points.

   (a) The set of all rational numbers with odd denominators (when written in lowest terms), along with 0.

   (b) The set of all rational numbers with even denominators (when written in lowest terms), along with 0.

   (c) The set of all rational numbers with odd numerators (when written in lowest terms), along with 0.

   (d) The set of squares of rational numbers.

2. Let R be a ring with $1 \neq 0$. The ***center*** of R is the set $\{z \in R \mid zr = rz \text{ for all } r \in R\}$.

   (a) Show that the center of R is a subring that contains the identity.

   (b) Show that the center of a division ring is a field.

   (c) Recall that the real Hamilton Quaternions is the ring

   $$\mathbb{H} = \{a + bi + cj + dk \mid a, b, c, d \in \mathbb{R}\},$$

   with component-wise addition, and multiplication as described in class. Describe the center of $\mathbb{H}$.

3. Let R be a ring with 1. An element $x \in R$ is called ***nilpotent*** if $x^n = 0$ for some positive integer n. Suppose R is commutative, and let x be a nilpotent element of R.

   (a) Show that x is either 0 or a zero divisor.

   (b) Prove $1 + x$ is a unit in R.

   (c) Show that the sum of a nilpotent element and a unit is a unit.

4. Let R and S be nonzero rings with identity, and $\varphi : R \to S$ a nonzero homomorphism.

   (a) Show that either $\varphi(1) = 1$ or $\varphi(1)$ is a zero divisor.

   (b) Show that if u is a unit and $\varphi(1) = 1$, then $\varphi(u)$ is a unit.

   (c) Show that if $a \in R$ and $\varphi(a)$ is nilpotent, then a is nilpotent or $\ker \varphi$ is nonzero.

5. (a) Show that the rings $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.

   (b) Show that the rings $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.