Math 122 - Problem Set 8
Due Wednesday, Nov 6

1. Let G be a group, and $M \leq G$ a maximal subgroup.
   (a) Show that $N_G(M) = M$ or $N_G(M) = G$.
   (b) Show that if M is not normal in G, then the number of nonidentity elements contained in conjugates of M is at most $(|M| - 1)|G : M|$.

2. Let G be a group.
   (a) Show that if $|G| = 200$, G has a normal Sylow 5-subgroup.
   (b) Show that if $|G| = 105$, G has a normal Sylow 5-subgroup or a normal Sylow 7-subgroup. *(Extra credit: Show it has BOTH a normal Sylow 5-subgroup and a normal Sylow 7-subgroup.)*
   (c) Show that if $|G| = 56$, G has a normal Sylow p-subgroup for some prime p dividing its order.

3. Let G be a finite group, and let p be the smallest prime dividing G. Let P be a Sylow p-subgroup. Suppose P has order p, with generator x. Let $H = N_G(P)$.
   (a) Show that P is the unique Sylow p-subgroup of H.
   (b) Show that $C_G(P) = C_H(P) = C_H(x)$.
   (c) Show that $|H : C_H(x)| < p$.
   (d) Conclude that $C_G(P) = N_G(P)$.

4. Let R be a ring with 1, and let S be a subring containing the identity.
   (a) Show that if u is a unit in S, then u is a unit in R.
   (b) Give an example to show that the converse of (a) is false.