Math 122 - Problem Set 7
Due Wednesday, Oct 23

1. Let G be a group and A a nonempty set that G acts on.

   (a) Show that if $a, b \in A$ and $b = g \cdot a$ for some $g \in G$, then $G_b = gG_a g^{-1}$.

   (b) Show that if G acts transitively on A then the kernel of the action is
   $$\bigcap_{g \in G} gG_a g^{-1}.$$

   (c) Assume that G is an abelian subgroup of S_A and the action on A is given by $\sigma \cdot a = \sigma(a)$ for $\sigma \in G$. Assume the action is transitive. Show that $\sigma(a) \neq a$ for all $\sigma \in G - \{1\}$ and $a \in A$.

   (d) Show that for G in part (c), $|G| = |A|$. *[Hint: First show that given an element $a \in A$ and $\sigma, \tau \in G$, if $\sigma(a) = \tau(a)$, then $\sigma = \tau$. Use this to construct a bijection $G \to A$.]*

2. Let Q_8 act on itself by left multiplication. Use this action to find a subgroup H of S_8 that is isomorphic to Q_8. You can just give generators for H – you don't need to list all the elements. (It might help to label the elements $1, -1, i, -i, j, -j, k, -k$ of Q_8 by $1, 2, 3, 4, 5, 6, 7, 8$ respectively.)

3. Let G be a non-cyclic group of order 6.

   (a) Show that each nontrivial element of G has order 2 or 3.

   (b) Show that the nontrivial elements can't all have the same order, and thus G has an element x of order 2 and y of order 3.

   (c) Show that if $xy = yx$, then $G = \langle xy \rangle$. Conclude that $xy \neq yx$.

   (d) Use part (c) to show that $\langle x \rangle$ is not normal.

   (e) Consider the action of G on the set of left cosets A of $\langle x \rangle$. Let $\pi_H : G \to S_A$ be the associated permutation representation. Show that $\ker \pi_H = 1$. (It might help to use a theorem from class.)

   (f) Conclude that the only two groups of order 6 (up to isomorphism) are Z_6 and S_3.

4. Find all conjugacy classes and their sizes in the following groups. (You might want to wait until after class on Monday to work on this problem.)

   (a) D_8

   (b) Q_8

   (c) A_4