Math 122 - Problem Set 6
Due Wednesday, Oct 16

1. Let H and K be subgroups of finite index in the group G, with $|G:H| = m$ and $|G:K| = n$.

   (a) Let k be the least common multiple of m and n. Show that
   $$k \le |G : H \cap K| \le mn.$$

   (b) Deduce that if m and n are relatively prime, then $|G : H \cap K| = mn$.

   (c) If $H \le K \le G$, show that $|G:H| = |G:K||K:H|$.

2. Let G be a group and $N \trianglelefteq G$. Recall that the Fourth Isomorphism Theorem says that there is a bijection between the subgroups of G that contain N and the subgroups of G/N. In particular, any subgroup of G/N is of the form $\bar{H} = H/N$ for some $H \le G$ containing N.

   Let A and B be subgroups of G that contain N. Prove the following.

   (a) $A \le B$ if and only if $\bar{A} \le \bar{B}$.

   (b) $A \trianglelefteq G$ if and only if $\bar{A} \trianglelefteq \bar{G}$.

   (c) If $A \le B$, then $|B:A| = |\bar{B} : \bar{A}|$.

3. Let M and N be normal subgroups of G such that $G = MN$.

   (a) Show that for any elements $m \in M$ and $n \in N$, there are elements $m' \in M$ and $n' \in N$ such that $mn' = nm'$.

   (b) Prove that
   $$G/(M \cap N) \cong (G/M) \times (G/N).$$
   [Hint: Come up with a map $G \to (G/M) \times (G/N)$ and show it's surjective. What is its kernel?]

4. (a) If G is a group and $H \le G$, show that for any $g \in G$, gHg^{-1} is a subgroup of G and $H \cong gHg^{-1}$

   (b) Let n be a nonnegative integer. Show that if G has exactly one subgroup H of order n, then $H \trianglelefteq G$.

   (c) Show that A_4 has exactly one subgroup of order 4, and that it is normal and isomorphic to $Z_2 \times Z_2$.