Math 122 - Problem Set 5
Due Wednesday, Oct 9

1. Let G be a group and $N \leq G$. Show that the following are equivalent.

   (i) $N \trianglelefteq G$

   (ii) $N_G(N) = G$

   (iii) $gN = Ng$ for all $g \in G$

   (iv) $gNg^{-1} \subseteq N$ for all $g \in G$.

2. Let $H \leq G$. Prove that the function $f : G \to G$ defined $f(x) = x^{-1}$ sends each left coset of H to a right coset of H and gives a bijection between the left cosets and right cosets. (Thus the number of left cosets equal the number of right cosets.)

3. Let $\phi : G \to H$ be a homomorphism.

   (a) Let $K \leq H$. Show that $\phi^{-1}(K) \leq G$.

   (b) If $K \trianglelefteq H$, show that $\phi^{-1}(K) \trianglelefteq G$.

   (c) Give an example to show that $L \trianglelefteq G$ doesn't necessarily imply $\phi(L) \trianglelefteq H$.

4. Let $G = Z_4 \times Z_4$, which has the following presentation:

$$G = \langle x, y \mid x^4 = y^4 = 1, xy = yx \rangle.$$

   Let $\bar{G} = G/\langle x^2y^2 \rangle$ (note that every subgroup of an abelian group is normal). For $g \in G$, denote the coset $g\langle x^2y^2 \rangle$ by $\bar{g}$.

   (a) Show by Lagrange's Theorem that $|\bar{G}| = 8$.

   (b) Write each element of $\bar{G}$ in the form $\bar{x}^a\bar{y}^b$ for some integers a and b.

   (c) Find the order of each of the elements of $\bar{G}$.

   (d) Show that $\bar{G} \cong Z_4 \times Z_2$.

5. Let G be a group. We showed in class that $Z(G) \trianglelefteq G$.

   (a) Show that if $G/Z(G)$ is cyclic, then G is abelian. *[Hint: Let $xZ(G)$ be a generator. Then every element of G can be written in the form x^az for some $a \in \mathbb{Z}$ and $z \in Z(G)$.]*

   (b) Show that if $|G| = pq$ for some primes p and q (not necessarily distinct), then either G is abelian or $Z(G) = 1$.