Math 122 - Problem Set 4
Due Wednesday, Oct 2

1. Let G be a finite group.
   (a) If $|G| = 3$, show that G is cyclic.
   (b) If $|G| = 4$, show that G is either cyclic, or isomorphic to $Z_2 \times Z_2$, the product of two cyclic groups of order 2.
   (c) Show that if G has prime order, it must be cyclic.
   (d) Show that G has even order if and only if it has an element of order 2. [Hint: Consider the set $\{x \in G \mid x \neq x^{-1}\}$. How many elements does it have? What is its complement?]

2. Let G be a group. An isomorphism from G to itself is called an ***automorphism*** of G. Let $\mathrm{Aut}(G)$ be the set of automorphisms of G.
   (a) Show that $\mathrm{Aut}(G)$ is a group with composition as the operation.
   (b) Show that $\mathrm{Aut}(G)$ acts on G by $\phi \cdot g = \phi(g)$, and the action has trivial kernel.
   (c) **Challenge:** Find the orbits of the action of $\mathrm{Aut}(S_3)$ on S_3.

3. Let n be a positive integer greater than 1. For each $a \in \mathbb{Z}$, define the homomorphism $\sigma_a : Z_n \to Z_n$ by $\sigma_a(x) = x^a$.
   (a) Show that σ_a is an automorphism if and only if a and n are relatively prime.
   (b) Show that $\sigma_a = \sigma_b$ if and only if $a \equiv b \pmod{n}$.
   (c) Prove that every automorphism of Z_n is equal to σ_a for some integer a.
   (d) Prove that $\sigma_a \circ \sigma_b = \sigma_{ab}$.
   (e) Deduce that the map $\bar{a} \to \sigma_a$ is an isomorphism of $(\mathbb{Z}/n\mathbb{Z})^\times$ onto the automorphism group of Z_n.

4. A group G is ***finitely generated*** if there is a finite set A such that $G = \langle A \rangle$.
   (a) Show that every finitely generated subgroup of $\mathbb{Q}$ (under addition) is cyclic.
   (b) Show that $\mathbb{Q}$ is not finitely generated.
   (c) Find a proper subgroup of $\mathbb{Q}$ (i.e. $H \leq \mathbb{Q}$ such that $H \neq \mathbb{Q}$) that is not cyclic.

5. A subgroup M of a group G is a ***maximal subgroup*** if $M \neq G$ and the only subgroups of G that contain M are M and G.
   (a) Show that if H is a proper subgroup of a finite group G, then there is a maximal subgroup of G containing H. (In fact, this is true in any finitely generated group – the proof uses Zorn's Lemma.)
   (b) Show that the subgroup of all rotations in a dihedral group is a maximal subgroup.
   (c) Show that if $G = \langle x \rangle$ is a cyclic group of order n, then a subgroup H is maximal if and only if $H = \langle x^p \rangle$ for some prime p dividing n.