Math 122 - Problem Set 3
Due Wednesday, Sept 25

1. Let H be a group acting on a set A. Consider the relation $\sim$ on A defined by $a \sim b$ if and only if $a = h \cdot b$ for some $h \in H$.

   (a) Show that $\sim$ is an equivalence relation, i.e. it's reflexive ($a \sim a$), symmetric ($a \sim b \implies b \sim a$) and transitive ($a \sim b$ and $b \sim c \implies a \sim c$). (For each $x \in A$ the equivalence class of x is called the ***orbit*** of x under the action of H. The orbits partition A).

   (b) Let G be a group and $H \leq G$ a subgroup of G. Check that H acts on G by left multiplication.

   (c) Assume G in part (b) is finite. Let $x \in G$ and let $\mathcal{O}$ be the orbit of x under the action of H. Prove that the map $f : H \to \mathcal{O}$ defined $f(h) = hx$ is a bijection.

   (d) Deduce that $|H|$ divides $|G|$. This is called *Lagrange's Theorem*.

2. (a) Consider the groups

   - D_8
   - Q_8
   - $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
   - $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$

   Notice that each of these groups has order 8. For each pair, show that the two groups are not isomorphic.

   (b) Is $S_3 \cong D_6$?

   (c) For both S_3 and Q_8, compute the centralizers of each element and find the center of each group.

3. Let G be an abelian group.

   (a) Show that $\{g \in G \mid |g| < \infty\} \leq G$. This is called the ***torsion subgroup*** of G.

   (b) Give an example to show the set in part (a) is not a subgroup if we don't require G to be abelian.

   (c) Fix $n > 1$ a positive integer. Find the torsion subgroup of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}$.

   (d) Show that the set of elements of $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}$ of infinite order together with the identity is not a subgroup.

4. Let G be a group. Show that the intersection of an arbitrary (not necessarily finite) collection of subgroups of G is again a subgroup.

5. Let G be a group and $H \leq G$.

   (a) Show $H \leq N_G(H)$.

   (b) Show $H \leq C_G(H)$ if and only if H is abelian.

(c) Let $A \subseteq G$ be a nonempty subset (not necessarily a subgroup). Define

$$N_H(A) = \{h \in H \mid hAh^{-1} = A\}.$$

Show that $N_H(A) = N_G(A) \cap H$, and thus $N_H(A)$ is a subgroup of H.