

Math 122 - Problem Set 2
Due Wednesday, Sept 18

1. Recall from class that D_{2n} is the dihedral group, with presentation

$$D_{2n} = \langle r, s \mid r^n = s^2 = 1, rs = sr^{-1} \rangle$$

- (a) If $n = 2k$ is even and $n \geq 4$, show that r^k is the only nonidentity element of D_{2n} that commutes with all elements of D_{2n} .
 - (b) If n is odd and $n \geq 3$, show that the identity is the only element that commutes with all elements of D_{2n} .
 - (c) For $n = 1$ and 2 , show that D_{2n} is abelian. To which familiar groups are these isomorphic?
2. (a) If G is a group, and $a, b \in G$ commute, show that $(ab)^n = a^n b^n$ for all $n \in \mathbb{Z}$. (Do this first by induction for positive n .)
- (b) Show that the order of a cycle in S_n is equal to its length.
 - (c) Show that the order of an element in S_n is equal to the least common multiple of the lengths of the cycles in its cycle decomposition.
 - (d) For each element in S_3 , write its cycle decomposition and compute its order.
3. Let G be a group.
- (a) For $x \in G$, show that $\{x^n \mid n \in \mathbb{Z}\}$ is a subgroup of G . This is called the **cyclic subgroup** of G generated by x .
 - (b) Find all the cyclic subgroups of D_8 .
 - (c) A group G is **cyclic** if G is a cyclic subgroup of itself; i.e. if $G = \{x^n \mid n \in \mathbb{Z}\}$ for some $x \in G$. Check that $\mathbb{Z}/n\mathbb{Z}$ and \mathbb{Z} are cyclic.
 - (d) Is $(\mathbb{Z}/n\mathbb{Z})^\times$ always cyclic?
4. Let $\phi : G \rightarrow G'$ be a group homomorphism.

- (a) Show that $H \subset G$ is a subgroup if and only if H is nonempty and for all $a, b \in H$, $a^{-1} \in H$ and $ab \in H$.
- (b) Define the **kernel** of ϕ to be

$$\ker(\phi) := \{g \in G \mid \phi(g) = 1\}.$$

Show that $\ker(\phi) \leq G$.

- (c) Show that ϕ is injective if and only if $\ker(\phi) = \{1\}$, the trivial subgroup.
- (d) Show that the image of ϕ is a subgroup of G' .
- (e) Show that the only homomorphism from $\mathbb{Z}/n\mathbb{Z}$ to \mathbb{Z} is the trivial homomorphism, i.e. the map sending all elements to the identity.