Math 122 - Problem Set 11 (Extra credit)
Due TUESDAY, Dec 3

Assume all rings have identity $1 \neq 0$. Very little partial credit will be given on these problems.

1. Let R be a commutative ring. Let $a, b \in R$ be nonzero. A ***least common multiple*** of a and b is an element $e \in R$ such that

   i. $a|e$ and $b|e$, and

   ii. if $a|e'$ and $b|e'$, then $e|e'$.

   (a) Show that if a least common multiple of a and b exists, then there is a unique largest principal ideal contained in $(a) \cap (b)$ and it is generated by e.

   (b) Show that any two elements of a UFD have a least common multiple, which is unique up to multiplication by a unit. Describe the least common multiple of a and b in terms of their prime factorizations.

   (c) Show that in a UFD, the least common multiple of a and b is $\frac{ab}{(a,b)}$, where (a, b) is the greatest common divisor of a and b.

2. Let R be a commutative ring. Let $I = (a_1, \ldots, a_n)$ and $J = (b_1, \ldots, b_m)$ be ideals in R. Show that the ideal IJ is generated by the elements of the form $a_i b_j$, where $1 \leq i \leq n$ and $1 \leq j \leq m$.

3. Let $R = \mathbb{Z}[\sqrt{-5}]$. Define the ideals $I = (2, 1 + \sqrt{-5})$, $J = (3, 2 + \sqrt{-5})$, and $K = (3, 2 - \sqrt{-5})$.

   (a) Show that I, J, and K are not principal.

   (b) Show that I, J, and K are prime.

   (c) Show $I^2 = (2)$.

   (d) Show $IJ = (1 - \sqrt{-5})$.

   (e) Show $IK = (1 + \sqrt{-5})$.

   (f) Conclude that $I^2JK = (6)$.

4. Let $R = \mathbb{Q}[x, y]$.

   (a) Show that (x) is prime.

   (b) Show that (x, y) is maximal (and thus prime).

   (c) Show that (x, y) is not principal.

   (d) Show that $(y^2 - x)$ is prime.

   (e) Conclude that $R/(y^2 - x)$ is not isomorphic to $R/(y^2 - x^2)$.

5. Let R be a commutative ring, and $I \subset R$ and ideal. Define the ***radical*** of I to be

$$\operatorname{rad}(I) = \{x \in R \mid x^n \in I \text{ for some } n \geq 1\}.$$

(a) Show that $\operatorname{rad}(I)$ is an ideal containing I.

(b) Show that $\operatorname{rad}(I)$ is contained in the intersection of the primes containing I.

(c) Show that $\operatorname{rad}(0)$ is the set of nilpotents of R.