Math 122 - Problem Set 10
Due Wednesday, Nov 20

Assume all rings have identity $1 \neq 0$.

1. Let R and S be commutative rings, and $\phi : R \to S$ a ring homomorphism such that $\phi(1) = 1$.

   (a) If $I \subset S$ is an ideal of S, show $\phi^{-1}(I)$ is an ideal of R. (In fact, this holds even for R and S noncommutative.)

   (b) Show that if $P \subset S$ is a prime ideal, then $\phi^{-1}(P)$ is a prime ideal of R.

   (c) Show that if ϕ is surjective and $M \subset S$ is a maximal ideal, then $\phi^{-1}(M)$ is maximal.

   (d) Give an example to show that (c) doesn't necessarily hold if ϕ is not surjective.

2. Let S be an integral domain and $f(x)$ a monic polynomial in $S[x]$ of degree $n \geq 1$ (i.e. the coefficient of x^n is 1). Define $R = S[x]/(f(x))$. Use bar notation to denote passage to the quotient; that is, if $g(x) \in S[x]$, then

   $$\overline{g(x)} := g(x) + (f(x)) \in R.$$

   (a) Show that every element of R is of the form $\overline{p(x)}$ for some polynomial $p(x) \in S[x]$ of degree less than n. [Hint: if $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then $\overline{f(x)} = \overline{0}$, so $\overline{x^n} = \overline{-a_{n-1}x^{n-1} - \cdots - a_0}$.]

   (b) Show that if $p(x)$ and $q(x)$ are distinct polynomials in $S[x]$, both of degree less than n, then $\overline{p(x)} \neq \overline{q(x)}$.

   (c) If $\overline{f(x)} = a(x)b(x)$ for some $a(x), b(x) \in S[x]$, both of degree less than n, show that $\overline{a(x)}$ is a zero divisor.

   (d) If $f(x) = a(x)b(x) - 1$ for some $a(x), b(x) \in S[x]$, show that $\overline{a(x)}$ is a unit.

3. Let $R = (\mathbb{Z}/2\mathbb{Z})[x]$.

   (a) Use 2. to show that $R/(x^2 + x + 1)$ has exactly four elements: $\overline{0}, \overline{1}, \overline{x}$, and $\overline{1 + x}$.

   (b) Show that $R/(x^2 + x + 1)$ is a field, and thus $(x^2 + x + 1) \subset R$ is a maximal ideal.

4. Let $S = \mathbb{R}[x]/(x^2 + 1)$.

   (a) Use 2. to conclude that $S = \{\overline{a + bx} \mid a, b \in \mathbb{R}\}$.

   (b) If $a_1, a_2, b_1, b_2 \in \mathbb{R}$, compute the product $(\overline{a_0 + a_1 x})(\overline{b_0 + b_1 x})$ in S. Write your answer in the form $\overline{c_0 + c_1 x}$.

   (c) Define the function $\phi : S \to \mathbb{C}$ by $\phi(\overline{a + bx}) = a + bi$. Show that this is an isomorphism! (Isn't that cool? Basically what we did here was force $x^2 + 1 = 0$, which means $x^2 = -1$. That is, we gave -1 a square root: x.)