Math 122 - Problem Set 1
Due Wednesday, Sept 11

1. Determine whether X along with the given binary operation $*$ is a group. If it is a group, show that all of the group axioms hold. If not, give a counterexample for one of the axioms.

   (a) X is the set of functions from $\mathbb{R}$ to $\mathbb{R}$, and define $f * g = f \cdot g$ (multiplication).

   (b) X is the set of functions from $\mathbb{R}$ to $\mathbb{R}$, and define $f * g = f \circ g$.

   (c) $X = \mathbb{R}_+$ and $a * b = \sqrt{ab}$.

   (d) $X = \{a/b \mid a \in \mathbb{Z} \text{ and } b \in \{1, 2\}\}$ and define $x * y = x + y$.

2. Let $n > 1$. Recall that $\mathbb{Z}/n\mathbb{Z}$ is the set of residue classes mod n.

   (a) Check that multiplication in $\mathbb{Z}/n\mathbb{Z}$ is well-defined; that is, if $\bar{a_1} = \bar{b_1}$ and $\bar{a_2} = \bar{b_2}$, then $\overline{a_1 \cdot a_2} = \overline{b_1 \cdot b_2}$.

   (b) Show that $\mathbb{Z}/n\mathbb{Z}$ is not a group under multiplication.

   (c) We define

   $$(\mathbb{Z}/n\mathbb{Z})^\times = \{\bar{a} \in \mathbb{Z}/n\mathbb{Z} \mid \text{there is } \bar{c} \in \mathbb{Z}/n\mathbb{Z} \text{ such that } \bar{a} \cdot \bar{c} = \bar{1}\}.$$

   Check that $(\mathbb{Z}/n\mathbb{Z})^\times$ is a group under multiplication. (You can assume that the standard multiplication on $\mathbb{Z}$ is associative.)

   (d) Show that $(\mathbb{Z}/n\mathbb{Z})^\times$ is the collection of residue classes whose representatives are relatively prime to n.

   (e) Write down the multiplication table for $(\mathbb{Z}/12\mathbb{Z})^\times$. What are the orders of each element?

3. Let G be a group, and $a, b \in \mathbb{Z}$. Let $x \in G$

   (a) Show $(x^a)^{-1} = x^{-a}$

   (b) Show $x^{a+b} = x^a x^b$ and $(x^a)^b = x^{ab}$.

   (c) Show that $|x| = |x^{-1}|$.

4. Let $\langle A, *_A \rangle$ and $\langle B, *_B \rangle$ be groups. Define the binary operation $*$ on $A \times B$ by $(a, b) * (c, d) = (a *_A c, b *_B d)$.

   (a) Show that $\langle A \times B, * \rangle$ is a group. The group $A \times B$ is called the ***direct product*** of A and B.

   (b) Show that $A \times B$ is abelian if and only if A and B are abelian.

5. Let G be a group and $x \in G$.

   (a) If x has finite order n, show that the elements $1, x, x^2, \ldots, x^{n-1}$ are all distinct.

   (b) If x has infinite order, show that the elements x^n, $n \in \mathbb{Z}$ are all distinct.