Math 101 - Problem Set 9
Due Tuesday, November 21

1. Determine whether the following subsets of $\mathbb{R}$ (given the standard topology) are open or not. Justify your answer.

   (a) $\mathbb{R} - \mathbb{Z}$.

   (b) $\mathbb{Z}$

   (c) $\displaystyle\bigcup_{0<a<b<1} [a, b]$

   (d) $\left\{ \frac{1}{a} \mid a \in \mathbb{Z}_+ \right\}$

2. Let $\mathcal{T}_1$ be the standard topology on $\mathbb{R}$ and $\mathcal{T}_2$ the cofinite topology on $\mathbb{R}$.

   (a) Show that if $U \in \mathcal{T}_2$, then $U \in \mathcal{T}_1$. i.e. $\mathcal{T}_2 \subseteq \mathcal{T}_1$.
   (We say in this situation that $\mathcal{T}_1$ is **finer** than $\mathcal{T}_2$ and that $\mathcal{T}_2$ is **coarser** than $\mathcal{T}_1$.)

   (b) Give an example to show that the reverse inclusion doesn't hold.

3. Determine whether each of the following collections of subsets is a basis for a topology on $\mathbb{R}$. Justify your answer. If it is a basis, describe the topology it generates.

   (a) $\mathcal{B} = \{B \subset \mathbb{R} \mid B \text{ is finite}\}$

   (b) $\mathcal{B} = \{B \subset \mathbb{R} \mid B \text{ is infinite}\}$

   (c) $\mathcal{B} = \{(a, b) \mid a < b \text{ and } a, b \in \mathbb{Z}\}$

   (d) $\mathcal{B} = \{\mathbb{R} - \{x\} \mid x \in \mathbb{R}\}$

   (e) $\mathcal{B} = \{(a - 1, a + 1) \mid a \in \mathbb{R}\}$

4. Let X be a set. Let $\mathcal{T}$ be the collection of all subsets U of X such that $X - U$ is either countable or is all of X. Show that $\mathcal{T}$ is a topology on X.

5. Let X be a topological space. Let $A \subseteq X$. Suppose that for each $x \in A$, there is an open set U containing x such that $U \subseteq A$. Show that A is open in X.

6. Consider the set $X = \{a, b, c, d\}$. Come up with 8 topologies on X that are different from each other even after permuting the elements. You can just circle the open sets like we did in class. You don't need to justify your answer. *(Note that there are actually more possibilities than 8–you don't have to list the rest of them!)*