Math 101 - Problem Set 6
Due Tuesday, October 24

1. Determine whether X along with the given binary operation $*$ is a group. If it is a group, show that all of the group axioms hold. If not, give a counterexample for one of the axioms.

    (a) X is the set of functions from $\mathbb{R}$ to $\mathbb{R}$, and define $f * g = f \cdot g$ (multiplication).

    (b) X is the set of functions from $\mathbb{R}$ to $\mathbb{R}$, and define $f * g = f \circ g$.

    (c) $X = \mathbb{R}_+$ and $a * b = \sqrt{ab}$.

    (d) $X = \{a/b \mid a \in \mathbb{Z} \text{ and } b \in \{1, 2\}\}$ and define $x * y = x + y$.

2. Find the order of each element of the group $\mathbb{Z}_{12}$.

3. Let G be a group and suppose there is an element $a \in G$ of finite order n.

    (a) Show that $e, a^1, \ldots, a^{n-1}$ are all distinct elements of G. (Recall that $a^i = a * a * \cdots * a$ for i factors a.)

    (b) If G is finite, show that $|a| \leq |G|$.

4. Let $G = \{e, a, b, c\}$. Come up with a binary operation $*$ on G so that G is a group with identity e and no elements have order 4. Write out the $*$ table and show that you can't rename the elements to get $\langle \mathbb{Z}_4, + \rangle$.

5. Let $G = \{a + b\sqrt{2} \in \mathbb{R} \mid a, b \in \mathbb{Q}\}$.

    (a) Show that G is a group under addition.

    (b) Show that $G - \{0\}$ is a group under multiplication.

6. Let $\langle A, *_A \rangle$ and $\langle B, *_B \rangle$ be groups. Define the binary operation $*$ on $A \times B$ by $(a, b) * (c, d) = (a *_A c, b *_B d)$. Show that $\langle A \times B, * \rangle$ is a group. The group $A \times B$ is called the ***direct product*** of A and B.

7. Show that $A \times B$ is abelian if and only if A and B are abelian.

8. Let G be an abelian group and let $a, b \in G$ and $n \in \mathbb{Z}_+$. Show that $(a * b)^n = a^n * b^n$. *(Hint: Induction!)*

9. Let G be a group, and suppose $a^2 = e$ for all $a \in G$. Show that G is abelian.