Math 101 - Problem Set 4
Due Tuesday, October 3

This problem set is slightly shorter so that you have extra time to study. The first exam will be on Thursday, October 5 in class.

1. Complete the proof of the theorem from class that if an ordered set A has the least upper bound property, then it has the greatest lower bound property by proving the following claim:

   Let A_0 be a nonempty subset of A that is bounded below. Let B be the set of lower bounds of A_0, and let $b_0 = \sup(B)$. Then $b_0 = \inf(A_0)$.

2. Let A be an ordered set. Suppose $a, b \in A$ and $a < b$. Consider the open interval

   $$(a, b) = \{x \mid a < x < b\}.$$

   If this set is empty, we call a the ***immediate predecessor*** of b and b the ***immediate successor*** of a. For example, in $\mathbb{Z}$, 1 is the immediate successor of 0. In $\mathbb{R}$, no element has an immediate successor or predecessor.

   Show that any element in an ordered set has at most one immediate successor.

3. For each of the following sets, given the dictionary order, give a counterexample to show that they don't satisfy the least upper bound property. Make sure to prove that your example works.

   (a) $[-1, 1] \times ([-1, 1] - \{0\})$

   (b) $[0, 1] \times (0, 1]$

4. Let $\mathbb{Z}_+$ denote the set of positive integers. Consider the following strict total order relations on $\mathbb{Z}_+ \times \mathbb{Z}_+$:

   - The dictionary order.
   - $(a, b) < (x, y)$ if either $a - b < x - y$ or $a - b = x - y$ and $b < y$.
   - $(a, b) < (x, y)$ if either $a + b < x + y$ or $a + b = x + y$ and $b < y$.

   In each of these relations, which elements have immediate predecessors? Does the set have a smallest element?

5. Let $f : (-1, 1) \to \mathbb{R}$ be defined

   $$f(x) = \frac{x}{1 - x^2}.$$

   Show that $(-1, 1)$ and $\mathbb{R}$ have the same order type by proving the following.

   (a) f is a bijection.

   (b) f is ***order-preserving***. That is, if $a, b \in (-1, 1)$ and $a < b$, then $f(a) < f(b)$.

6. Show that the subset of $\mathbb{R}$, $(0, 1) \cup \{2\}$, has the same order type as $(1, 2]$.

7. Show that there is no function $f : \mathbb{R} \to \mathbb{Z}$ such that if $a, b \in \mathbb{R}$ and $a < b$, then $f(a) < f(b)$.