Math 101 - Problem Set 3
Due Tuesday, Sept 26

1. Let $f : A \to B$ and $A_0 \subseteq A$ and $B_0 \subseteq B$.

   (a) Show that if f is surjective, then $f(f^{-1}(B_0)) = B_0$.

   (b) Show that if f is injective, then $f^{-1}(f(A_0)) = A_0$.

2. Let $f : A \to B$ and $A_0, A_1 \subseteq A$. Prove the following.

   (a) $f(A_0 \cup A_1) = f(A_0) \cup f(A_1)$.

   (b) $f(A_0 \cap A_1) \subseteq f(A_0) \cap f(A_1)$.

   (c) If f is injective, equality holds in part (b).

3. Let $f : A \to B$ and $g : B \to C$ be injective functions. Show that $g \circ f$ is injective.

4. Let $f : X \to Y$ be a surjective function. Define the equivalence relation $\sim$ on X as follows:

   $$a \sim b \iff f(a) = f(b).$$

   Recall the function $g : Y \to X/\sim$ we defined in class as

   $$g(y) = \{x \in X \mid f(x) = y\}.$$

   Show g is injective (in a different way than we proved it in class) by assuming $g(y_1) = g(y_2)$ and concluding $y_1 = y_2$.

5. Show that $h : X \to Y$ is surjective if and only if $h^{-1}(Y_0) \neq \emptyset$ for all nonempty $Y_0 \subseteq Y$.

6. Denote the identity function for a set C by i_C. Given $f : A \to B$, a function $g : B \to A$ is a ***left inverse*** for f if $g \circ f = i_A$ and $h : B \to A$ is a ***right inverse*** for f if $f \circ h = i_B$.

   (a) Show that if f has a left inverse, f is injective.

   (b) Show that if f has a right inverse, f is surjective.

7. (a) Give an example of a function that has a left inverse but no right inverse.

   (b) Give an example of a function that has a right inverse but no left inverse.

   (c) Is it possible for a function to have more than one left inverse? More than one right inverse? Either give examples or prove that it's not possible.

8. Show that if f has both a left inverse g and a right inverse h then f is bijective and $g = h = f^{-1}$.

9. Define the relation $\sim$ on $\mathbb{R}^2$ by

   $$(a, b) \sim (c, d) \iff a - b^2 = c - d^2.$$

   Show that this is an equivalence relation and describe the equivalence classes.

10. Let $\mathcal{L}$ be the collection of all straight lines in the plane (i.e. $\mathbb{R}^2$). Find a subset of $\mathcal{L}$ that is a partition of $\mathbb{R}^2$ and describe (mathematically) the corresponding equivalence relation.