Math 101 - Problem Set 2
Due Tuesday, Sept 19

1. Let A and B be sets. Prove the following.

   (a) $A \cap (B - C) = (A \cap B) - (A \cap C)$

   (b) $A - (A - B) = A \cap B$

2. Given sets A, B, and C, express each of the following sets in terms of A, B, and C, using the symbols $\cup$, $\cap$, and $-$.

   (a) $D = \{x \mid x \in A \text{ and } (x \in B \text{ or } x \in C)\}$

   (b) $E = \{x \mid (x \in A \text{ and } x \in B) \text{ or } x \in C\}$

   (c) $F = \{x \mid x \in A \text{ and } (x \in B \implies x \in C)\}$

3. Let $\mathcal{C}$ be a nonempty collection of sets. Prove the following facts about the power set.

   (a) $\displaystyle\bigcap_{X \in \mathcal{C}} \mathscr{P}(X) = \mathscr{P}\left(\bigcap_{X \in \mathcal{C}} X\right)$

   (b) $\displaystyle\bigcup_{X \in \mathcal{C}} \mathscr{P}(X) \subseteq \mathscr{P}\left(\bigcup_{X \in \mathcal{C}} X\right)$

4. Complete the following statement. "If A and B are sets, then $\mathscr{P}(A) \cup \mathscr{P}(B) = \mathscr{P}(A \cup B)$ if and only if . . . " Prove your assertion.

5. Again let $\mathcal{C}$ be a nonempty collection of sets.

   (a) Show that $\mathcal{C} \subseteq \mathscr{P}\left(\displaystyle\bigcup_{X \in \mathcal{C}} X\right)$.

   (b) Give necessary and sufficient conditions under which equality holds in part (a) (i.e. equality holds if and only if. . .). Prove your assertion.

6. Let A, B, C, and D be sets.

   (a) Assume $A \subseteq C$ and $B \subseteq D$. Show that $A \times B \subseteq C \times D$.

   (b) Give an example to show that if $X \subseteq C \times D$, it is not necessarily the case that $X = E \times F$ where $E \subseteq C$ and $F \subseteq D$, respectively.

7. For each of the following subsets of $\mathbb{R} \times \mathbb{R}$, determine whether it is equal to the Cartesian product of two subsets of $\mathbb{R}$. If it is, express it as the Cartesian product of two subsets.

   (a) $\{(x, y) \mid x \in \mathbb{Z}\}$

   (b) $\{(x, y) \mid 0 < y \leq 1\}$

   (c) $\{(x, y) \mid y > x\}$

   (d) $\{(x, y) \mid x \notin \mathbb{Z} \text{ and } y \in \mathbb{Z}\}$

8. Let A, B, X, and Y be sets. Prove the following equalities.

   (a) $(A \cap B) \times (X \cap Y) = (A \times X) \cap (B \times Y)$

   (b) $(A - B) \times X = (A \times X) - (B \times X)$

9. For each of the three properties of equivalence relations (reflexivity, symmetry, and transitivity), come up with an example of a relation that does not have that property, but does have the other two.

10. Here is a "proof" that every relation C that is both symmetric and transitive is also reflexive: "Since C is symmetric, aCb implies bCa. Since C is transitive, aCb and bCa together imply aCa, as desired." Find the flaw in this argument.