Math 101 - Problem Set 10
Due Thursday, November 30

1. Consider the following topologies on $\mathbb{R}$:

   - $\mathcal{T}_1$ = the standard topology,
   - $\mathcal{T}_2$ = the cofinite topology,
   - $\mathcal{T}_3$ = the topology with basis $\{(-\infty, a) \mid a \in \mathbb{R}\}$,
   - $\mathcal{T}_4$ = the topology with basis $\{(a, b) \mid a < b \text{ and } a, b \in \mathbb{Z}\}$,
   - $\mathcal{T}_5$ = the lower limit topology (which has basis $\{[a, b) \mid a < b\}$).

   We already showed in class that $\mathcal{T}_5$ is finer than $\mathcal{T}_1$, i.e. $\mathcal{T}_1 \subseteq \mathcal{T}_5$. For the remaining pairs of topologies, either show that one is finer than the other or that they are not comparable. Be sure to justify your answer.

2. Let X be a set and $\mathcal{B}$ a basis. Let $\mathcal{T}$ be the topology generated by $\mathcal{B}$. Show that $\mathcal{T}$ is the smallest (i.e. coarsest) topology containing $\mathcal{B}$. That is, if $\mathcal{T}'$ is another topology on X and $\mathcal{B} \subseteq \mathcal{T}'$, then $\mathcal{T} \subseteq \mathcal{T}'$.

3. Consider the topology $\mathcal{T}_K$ on $\mathbb{R}$, defined in class, that is generated by the following basis:

   $$\mathcal{B} = \{(a, b) \mid a < b\} \cup \{(a, b) - K \mid a < b\},$$

   where $K = \{\frac{1}{n} \mid n \in \mathbb{Z}_+\}$.

   (a) Show that $\mathcal{T}_K$ and the lower limit topology are not comparable.

   (b) Are $\mathcal{T}_K$ and the upper limit topology comparable? If so, which is finer? (The upper limit topology has basis $\{(a, b] \mid a < b\}$.)

4. *I know we've talked about this before a few times, but there is still some confusion and I just want to make sure it's clear:*

   Let X be a set and $\mathcal{T}$ a topology on X. Suppose that if $x \in X$ then $\{x\} \in \mathcal{T}$. Show that $\mathcal{T}$ is the discrete topology.

   The next two problems depend on material that we will cover on Tuesday, November 28.

5. Show that the order topology on $\mathbb{Z}$ is the discrete topology.

6. Consider $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ given the order topology (with the dictionary order). In each case, show whether or not the given subset is open.

   (a) $U = [1, 2] \times (1, 2)$

   (b) $U = (1, 2) \times (1, 2)$

   (c) $U = (1, 2) \times [1, 2]$

   (d) $U = [0, 1) \times [0, \infty)$