Math 101 - Problem Set 1
Due Tuesday, Sept 12

1. Write the negation of the following statements and predicates. Assume that x is a real number.

   (a) All cows are brown.

   (b) $x \leq 2$.

   (c) All cows are brown and eat grass.

   (d) x is a prime number greater than 12.

   (e) There exists a cow that is black and doesn't eat grass.

   (f) Every square is a rectangle.

   (g) If x is not an integer, then x is negative.

   (h) Every nonzero integer is either negative or positive.

   (i) If x is an odd integer, then x^2 will be a positive odd integer.

   (j) If it is cold or raining, I am wearing long sleeves or a jacket.

2. (a) Come up with two statements, P and Q such that $P \implies Q$ is true but $Q \implies P$ is false.

   (b) Using your statements from (a), write out "If $\neg Q$, then $\neg P$." (*"If $\neg Q$, then $\neg P$" is called the **contrapositive** of "If P, then Q." Notice that these two statements are equivalent!*)

3. (a) Prove DeMorgan's second law:

$$\neg(P \vee Q) \iff (\neg P) \wedge (\neg Q).$$

   You can use either the method we used in class and show that if the left side is true, the right side is true and if the left side is false then the right side is false (why is this sufficient?) or use the fact that two statements are equivalent if and only if they imply each other.

   (b) Using a truth table, verify both of DeMorgan's laws. Recall that the first law claims

$$\neg(P \wedge Q) \iff (\neg P) \vee (\neg Q).$$

4. List all of the subsets of the following sets:

   (a) $\emptyset$

   (b) $\{1, 2, 3\}$

   (c) $\Big\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\Big\}$

5. Let A, B, and C be sets. Prove that $(A \cap B) \cup C = A \cap (B \cup C)$ if and only if $C \subseteq A$.

6. Let $\mathcal{C}$ be a collection of sets.

   (a) Prove

$$A \cup \left(\bigcap_{S \in \mathcal{C}} S \right) = \bigcap_{S \in \mathcal{C}} (A \cup S).$$

(b) Give a similar formula for

$$A \cap \left(\bigcup_{S \in \mathcal{C}} S \right)$$

and prove that your formula is correct.

7. For each collection of sets $\mathcal{C}$, calculate $\bigcup_{S \in \mathcal{C}} S$ and $\bigcap_{S \in \mathcal{C}} S$.

   (a) A_1 = the set of odd integers,
   A_2 = the set of positive real numbers,
   A_3 = the set of integers divisible by 3, and
   $\mathcal{C} = \{A_1, A_2, A_3\}$.

   (b) For each $i \in \mathbb{N}$, $S_i = \{x \in \mathbb{N} \mid x \geq i\}$, and $\mathcal{C} = \{S_i \mid i \in \mathbb{N}\}$.

   (c) For every $n \in \mathbb{Z}$, $T_n = \mathbb{Z} - \{n\}$, i.e. the set of integers other than n, and $\mathcal{C} = \{T_n \mid n \in \mathbb{Z}\}$.

   (d) T_n is defined as in part (c), and $\mathcal{C} = \{T_n \mid n \text{ is a prime number}\}$.

8. Prove DeMorgan's Laws for sets: $(A \cap B)' = A' \cup B'$ and $(A \cup B)' = A' \cap B'$.
   (We will define the complement of a set, A', on Thursday, Sept 7.)

9. Choose one of DeMorgan's Laws from problem 8 and formulate and prove it for arbitrary unions or intersections (i.e. instead of a pair of sets, formulate it in terms of an arbitrary collection of sets $\mathcal{C}$.)