1. [6 points]

a. [4 points] Let X be a finite set. Let $\mathcal{T}$ be a topology on X. Define

$$\mathcal{T}' = \{S \mid X - S \in \mathcal{T}\}.$$

Show that $\mathcal{T}'$ is a topology on X.

$X - \phi = X \in \mathcal{T}$ and $X - X = \phi \in \mathcal{T}$, so $\phi, X \in \mathcal{T}'$

Let $\mathcal{C}$ be a subset of $\mathcal{T}'$. Since $\mathcal{T}' \subseteq \mathcal{P}(X)$ and X is finite, $\mathcal{T}'$ is finite and thus so is $\mathcal{C}$.

Set $V = \bigcup_{S \in \mathcal{C}} S$. Then $X - V = X - \bigcup_{S \in \mathcal{C}} S$

$= \bigcap_{S \in \mathcal{C}} (X - S)$, a finite intersection.

Since $X - S \in \mathcal{T}$ $\forall$ $S \in \mathcal{C}$, $X - V$ must be in $\mathcal{T}$, so $V \in \mathcal{T}'$

Now let $S_1, S_2, \ldots, S_n \in \mathcal{T}'$.

Then $X - (S_1 \cap S_2 \cap \cdots \cap S_n) = (X - S_1) \cup (X - S_2) \cup \ldots \cup (X - S_n)$.

$X - S_i \in \mathcal{T}$ for each i, so this is the finite union of elements of $\mathcal{T}$ and is thus in $\mathcal{T}$.

Thus $S_1 \cap \cdots \cap S_n \in \mathcal{T}'$, so $\mathcal{T}'$ is a topology.

b. [2 points] Give a counterexample to show that the statement in part a. is false if X can be infinite.

Let $X = \mathbb{R}$ and $\mathcal{T}$ be the cofinite topology.

Then $\mathcal{T}' = \{S \mid S \text{ is finite}\} \cup \{\mathbb{R}\}$.

$\{n\} \in \mathcal{T}'$ $\forall$ $n \in \mathbb{Z}$, but

$\bigcup_{n \in \mathbb{Z}} \{n\} = \mathbb{Z} \notin \mathcal{T}'$, so $\mathcal{T}'$ is not a topology on $\mathbb{R}$.

2. [8 points] Consider $X = \mathbb{R} \times [0,1]$ with the dictionary order. Let $\mathcal{T}$ be the order topology on X and let $\mathcal{T}^*$ be the topology on X generated by the basis

$$\{(a,b) \times [0,1] \mid a < b\}.$$

(You don't need to show that this is a basis.)

a. [4 points] Show $\mathcal{T}$ is finer than $\mathcal{T}^*$.

We call the given basis $\mathcal{B}$.

Let $B \in \mathcal{B}$ where $B = (a,b) \times [0,1]$.

Since 1 is the largest element of $[0,1]$,

$a \times 1 < c \times d \iff a < c$.

Similarly, $c \times d < b \times 0 \iff c < b$.

Thus, $B = (a,b) \times [0,1] = (a \times 1, b \times 0)$,

which is in $\mathcal{T}$.

Thus, $\mathcal{B} \subseteq \mathcal{T}$, so by problem 2 on PS 10,

$\mathcal{T}^* \subseteq \mathcal{T}$.

b. [4 points] Show $\mathcal{T} \neq \mathcal{T}^*$. That is, show $\mathcal{T}^*$ is **not** finer than $\mathcal{T}$.

Consider the set $(0 \times 0, 0 \times 1) \in \mathcal{T}$.

Let $B \in \mathcal{B}$ s.t. $B = (a,b) \times [0,1]$ and $0 \times \frac{1}{2} \in B$.

Then $a < 0 < b$, so $a < 0 < \frac{b}{2} < b$.

Thus, $\frac{b}{2} \times 0 \in B$ but $\frac{b}{2} \times 0 \notin (0 \times 0, 0 \times 1)$

since $\frac{b}{2} \neq 0$.

So $B \not\subseteq (0 \times 0, 0 \times 1)$ so $(0 \times 0, 0 \times 1) \notin \mathcal{T}^*$.

Thus, $\mathcal{T} \not\subseteq \mathcal{T}^*$, so $\mathcal{T} \neq \mathcal{T}^*$.

3. [10 points] Give an example of each of the following. You do not need to prove that your example works.

a. [2 points] A subset of $\mathbb{R}$ that is neither open nor closed in the lower limit topology.

$(0, 1]$

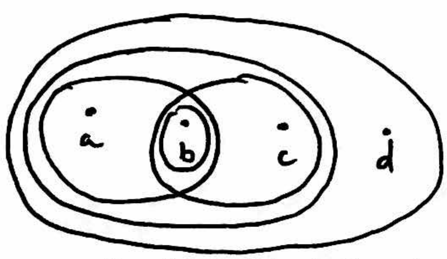
b. [2 points] A group G such that $|G| > 2$ and if $H \leq G$, then $H = \{e\}$ or G. i.e. G has no proper nontrivial subgroups.

$G = \mathbb{Z}_3$

c. [2 points] A set S along with an order relation such that every element has an immediate successor but at least one element (other than the smallest element) doesn't have an immediate predecessor.

$S = \mathbb{Z}_+ \times \mathbb{Z}_+$ with the dictionary order.

d. [2 points] A topology $\mathcal{T}$ on the set $X = \{a, b, c, d\}$ such that $\{a, b\}, \{b, c\} \in \mathcal{T}$, but $\mathcal{T}$ is not the discrete topology.



$\mathcal{T} = \{\emptyset, \{b\}, \{a,b\} \{b,c\}, \{a,b,c\}, X\}$

e. [2 points] A basis for a topology on $\mathbb{R}$ that is not comparable to the standard topology.

$\{(-\infty, 0), [0, \infty)\}$

4. [8 points] Let $\mathcal{T}$ be the standard topology on $\mathbb{R}$. Define

$$\mathcal{T}' = \mathcal{T} \cup \{U \cup \{0\} \mid U \in \mathcal{T}\}.$$

a. [4 points] Check that $\mathcal{T}'$ is a topology on $\mathbb{R}$.

Since $\phi, X \in \mathcal{T}$, $\phi, X \in \mathcal{T}'$ as well.

Let $\mathcal{C} \subseteq \mathcal{T}'$. If $S \in \mathcal{C}$, $S = U$ or $U \cup \{0\}$,
where $U \in \mathcal{T}$.

Thus if $V = \bigcup_{S \in \mathcal{C}} S$, then $V = W$ or $W \cup \{0\}$,
where W is the union of elements of $\mathcal{T}$ and is thus in $\mathcal{T}$. $\Rightarrow V \in \mathcal{T}'$

Now let $S_1, \dots, S_n \in \mathcal{T}'$. Then for each S_i, $\exists\, U_i \in \mathcal{T}$ s.t. $S_i = U_i$ or $U_i \cup \{0\}$.

Thus, $S_1 \cap S_2 \cap \dots \cap S_n = \bigcap_{i=1}^{n} U_i$ or $\left(\bigcap_{i=1}^{n} U_i\right) \cup \{0\}$.

Since $\bigcap_{i=1}^{n} U_i$ is the finite intersection of elements of $\mathcal{T}$, it's in $\mathcal{T}$, so $S_1 \cap \dots \cap S_n \in \mathcal{T}'$, so $\mathcal{T}'$ is a topology on $\mathbb{R}$.

b. [4 points] Consider $\mathbb{R}$ equipped with the topology $\mathcal{T}'$. Let $A \subseteq \mathbb{R}$. Show that $0 \in \bar{A}$ if and only if $0 \in A$.

If $0 \in A$, then $A \subseteq \bar{A}$, so $0 \in \bar{A}$.

On the other hand, assume $0 \in \bar{A}$.

Then since $\phi \in \mathcal{T}$, $\{0\} \cup \phi = \{0\} \in \mathcal{T}'$.

$\{0\} \cap A \neq \phi$ by a theorem in class,

So $0 \in A$.

5. [8 points] Determine whether the following statements are true or false, and briefly justify your answer (or give a counterexample, if applicable).

a. [2 points] Let $\mathcal{B}$ and $\mathcal{B}'$ be two bases on a set X and let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies they generate. If $\mathcal{B} \neq \mathcal{B}'$, then $\mathcal{T} \neq \mathcal{T}'$.

False.

On $\mathbb{R}$, $\{\{x\} \mid x \in \mathbb{R}\}$ and $\mathcal{P}(\mathbb{R})$ are both bases that generate the discrete topology.

b. [2 points] Let $\mathcal{B}$ be a basis generating a topology $\mathcal{T}$ on X. If $\mathcal{T}$ has uncountably many elements, then $\mathcal{B}$ does as well.

False.

Let $X = \mathbb{R}$ and $\mathcal{B} = \{\{x\} \mid x \in \mathbb{Z}\}$. Then $\mathcal{B}$ is countable, but it generates the discrete topology $\mathcal{T} = \mathcal{P}(\mathbb{Z})$, which is uncountable.

c. [2 points] If $f : G \to H$ is a surjective group homomorphism and H is abelian, then G is abelian.

False.

Let $G = D_8$, $H = \{e\}$. Define
$f : D_8 \to \{e\}$ by $f(x) = e \ \forall x \in D_8$.
H is abelian, but G isn't.

d. [2 points] If X is a set and $\sim$ an equivalence relation on X, then $X/\sim$ is a basis for a topology on X.

True.

Let $x \in X$. Then x is contained in the equivalence class $\{y \in X \mid x \sim y\} \in X/\sim$

Since every pair of equivalence classes is disjoint, no elements lie in the intersection.

Thus, $X/\sim$ is a basis.