1. [8 points] Determine whether the following statements are true or false, and briefly justify your answer (or give a counterexample, if applicable).

a. [2 points] If G and H are groups and $f : G \to H$ is an injective homomorphism that is not surjective, then G and H are not isomorphic.

False.

Set $G = H = \mathbb{Z}$.

Define $f: \mathbb{Z} \to \mathbb{Z}$ by

$f(x) = 2x$.

f is injective, but not surjective, while $G \cong H$.

b. [2 points] The set of functions from $\{0, 1\}$ to $\mathbb{Z}_+$ is uncountable.

False.

There is a bijection between this set and $\mathbb{Z}_+ \times \mathbb{Z}_+$, which is a finite product of countable sets, and is thus countable.

c. [2 points] If $G = \langle a \rangle = \langle b \rangle$ is a finite cyclic group, then $a = b$.

False.

$\mathbb{Z}_4 = \langle 1 \rangle = \langle 3 \rangle$.

d. [2 points] The subgroup $\{e, r, r^2, r^3\}$ of D_8 is isomorphic to the subgroup $\{1, -1, i, -i\}$ of Q_8.

True. $\{e, r, r^2, r^3\} = \langle r \rangle$

and $\{1, -1, i, -i\} = \langle i \rangle$.

Thus, both are cyclic groups of order 4, so they must be isomorphic to $\mathbb{Z}_4$, and thus to each other.

2. [7 points] Let G be an abelian group. Show that for all $a, b \in G$, $(ab)^n = a^n b^n$ for all $n \in \mathbb{Z}_+$.

We prove this by induction.

The base case is $n=1$:

$$(ab)^1 = ab = a^1 b^1.$$

Now assume $(ab)^k = a^k b^k$.

Then
$$(ab)^{k+1} = (ab)^k (ab)$$
$$= a^k (b^k a) b$$
$$= a^k (a b^k) b$$
$$= a^{k+1} b^{k+1}.$$

Thus, $(ab)^n = a^n b^n$ $\forall\, n \in \mathbb{Z}_+$, as desired.

3. [8 points] Let A, B, and C be groups, and let $f : A \to B$ and $g : B \to C$ be homomorphisms. Define $\varphi = g \circ f$.

a. [2 points] Show that $\varphi : A \to C$ is a homomorphism.

Let $x, y \in A$.

Then $\varphi(xy) = g(f(xy))$

$= g(f(x)f(y))$

$= g(f(x))g(f(y))$

$= \varphi(x)\varphi(y)$, so

φ is a homomorphism.

b. [6 points] Assume g is injective. Show $\operatorname{Ker}(\varphi) = \operatorname{Ker}(f)$.

First let $x \in \ker(\varphi)$.

Then $\varphi(x) = e$

$\Rightarrow g(f(x)) = e = g(e)$.

Since g is ~~isomor~~ injective,

$e = f(x)$,

So $x \in \ker(f)$

$\Rightarrow \ker(\varphi) \subseteq \ker(f)$.

If $x \in \ker(f)$, then

$\varphi(x) = g(f(x))$

$= g(e)$

$= e$.

Thus, $x \in \ker(\varphi)$, so

$\ker(\varphi) = \ker(f)$.

4. [10 points] Give an example of each of the following. You do not need to prove that your example works.

a. [2 points] An infinite group G that has a non-identity element of finite order.

$G = \mathbb{Z} \times \mathbb{Z}_2$

b. [2 points] A nonempty set S such that S^ω is countable.

$S = \{0\}$

c. [2 points] A group G and a subgroup $H \leq G$ such that H is abelian but G is not and $|H| \geq 2$.

$G = D_8$, $H = \{e, s\}$

d. [2 points] An uncountable S set that has cardinality different from $\mathbb{R}$

$S = \mathcal{P}(\mathbb{R})$

e. [2 points] Groups G and H, a homomorphism $f : G \to H$, and an element $x \in G$ such that $|x| \neq |f(x)|$.

$G = \mathbb{Z}_2$, $H = \{e\}$

Define $f(0) = f(1) = e$.

Then $|1| = 2$, but $|f(1)| = 1$.

5. [7 points] Let G be a group and H and K subgroups of G. Show that if $H \cup K$ is a subgroup of G then $H \subseteq K$ or $K \subseteq H$.

[Hint: Assume neither set contains the other. Then we can take $x \in K - H$ and $y \in H - K$.]

We show this by proving the contrapositive.

Assume H is not a subset of K and K is not a subset of H.

Then, we can find $x \in K - H$ and $y \in H - K$.

Define ~~$z \in x$~~ $z = xy$.

If $z \in K$, then

$y = x^{-1}z \in K$. But $y \notin K$, so $z \notin K$.

Similarly, if $z \in H$, then

$x = zy^{-1} \in H$. But $x \in K$, so $z \notin H$.

Thus, $z = xy \notin H \cup K$, which means that $H \cup K$ is not closed under the group operation, and is thus not a subgroup.