1. [8 points] Short answer. For each part of this problem, you do not need to justify your answer.

a. [2 points] Write the negation of the following sentence: "If $x, y \in \mathbb{Z}$ such that $y < x$, then there is some $z \in \mathbb{Q}$ such that $y < z < x$."

$\exists\ x, y \in \mathbb{Z}$ where $y < x$ such that

$\forall\ z \in \mathbb{Q}$, $z \leq y$ or $x \leq z$.

b. [2 points] Let A and B be sets and $f : A \to B$ a function. Define the equivalence relation $\sim$ on A by $a_1 \sim a_2$ if and only if $f(a_1) = f(a_2)$. Let $g : B \to A/\sim$ be the function we defined in class. Complete the following sentence: "g is well-defined if and only if ..."

g is well-defined if and only if f is surjective.

c. [2 points] Give an example of a relation that is reflexive but is neither symmetric nor transitive.

let R be the relation on $\{0,1,2\}$ defined

$R = \{(0,0),(1,1),(2,2),(0,1),(1,2)\}$

or, let R be the relation on $\mathbb{Z}$ defined

$x R y \iff$

$x \leq y \leq x+1$.

d. [2 points] List the elements of $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset)))$, where $\emptyset$ is the empty set.

$$\mathscr{P}(\mathscr{P}(\mathscr{P}(\phi))) = \mathscr{P}(\mathscr{P}(\{\phi\}))$$

$$= \mathscr{P}(\{\phi, \{\phi\}\})$$

$$= \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$$

2. [9 points] Let A, B, and C be sets and $f : A \to B$ and $g : B \to C$.

a. [7 points] Show that if $g \circ f$ is injective and f is surjective, then g is injective.

If B is empty or has only one element, then g is injective. Thus, suppose

$b_1, b_2 \in B$ s.t. $b_1 \neq b_2$.

Since f is surjective, $\exists\, a_1, a_2 \in A$ s.t.

$f(a_1) = b_1$ and $f(a_2) = b_2$.

Since $b_1 \neq b_2$, we know $a_1 \neq a_2$.

Injectivity of $g \circ f$ implies

$$(g \circ f)(a_1) \neq (g \circ f)(a_2)$$

$$\Rightarrow g(b_1) = g(b_2),$$

So g is injective.

b. [2 points] Give an example of functions $g : B \to C$ and $f : A \to B$ where $g \circ f$ is injective, but g is not injective. Make sure to specify what A, B, and C are in your example. You do not need to justify your answer.

Let $A = C = \{1\}$ and $B = \{1, 2\}$.

Define f and g by

$f(1) = g(1) = g(2) = 1$. Then $(g \circ f) = id_{\{1\}}$ and is thus injective, while g is not.

<u>or</u> Define $A = \mathbb{R}_{\geq 0}$, $B = C = \mathbb{R}$, and

$f(x) = \sqrt{x}$, $g(y) = y^2$.

3. [7 points] Let A and B be ordered sets that have the same order type. Show that if $b \in B$ has an immediate predecessor, there is some element of A that also has an immediate predecessor.

Let $f: A \to B$ be an order-preserving bijection, and let b_0 be the immediate predecessor of b.

Then $(b_0, b) = \phi$ and $b_0 < b$.

Since f is surjective, $\exists\ a_0, a \in A$ s.t.

$$f(a_0) = b_0 \text{ and } f(a) = b.$$

Since it's not the case that $b \leq b_0$, we know it's not true that $a \leq a_0$. Thus,

$$a_0 < a.$$

Suppose $x \in (a_0, a)$ (for the sake of contradiction).

Then $a_0 < x < a$, so

$$b_0 = f(a_0) < f(x) < f(a) = b,$$

so $f(x) \in (b_0, b)$, which is a contradiction.

Thus, $(a_0, a) = \phi$, so a has an immediate predecessor, a_0.

4. [8 points] Show that if an ordered set X has the greatest lower bound property, then it has the least upper bound property. (This is the converse of a statement we proved in class.)

Assume X has the g.l.b. property.

Let $\emptyset \neq Y \subseteq X$ be a set that is bounded above, and let $B \subseteq X$ be the (nonempty) set of upper bounds of Y.

Let $y \in Y$. Then $y \leq b$ $\forall$ $b \in B$, so B is bounded below and thus has a greatest lower bound, $\inf(B)$.

Since every element of Y is a lower bound for B, $\inf(B) \geq y$ $\forall$ $y \in Y$, so $\inf(B)$ is an upper bound for Y.

On the other hand, $\inf(B) \leq b$ $\forall$ $b \in B$, so $\inf(B)$ is the least upper bound of Y.

Thus, X has the l.u.b. property, as desired.

5. [8 points] Determine whether the following statements are true or false, and briefly justify your answer (or give a counterexample, if applicable).

a. [2 points] The relation $\sim$ on $\mathbb{R} \times \mathbb{R}$ defined $(x, y) \sim (z, w)$ if and only if $z = ax$ and $w = ay$ for some $a \in \mathbb{R}$ is an equivalence relation.

False.

$(1,1) \sim (0,0)$ since $0 = 0 \cdot 1$

but $(0,0) \not\sim (1,1)$ since $1 \neq a \cdot 0$ for any $a \in \mathbb{R}$

b. [2 points] If $f : A \to B$ is a function, and $A_0 \subseteq A$, then $f(A - A_0) = f(A) - f(A_0)$.

False.

Let $A = \{0,1\}$, $B = \{0\}$, $A_0 = \{1\}$.

Define $f(0) = f(1) = 0$.

Then $f(A - A_0) = f(\{0\}) = \{0\}$, but

$f(A) - f(A_0) = \{0\} - \{0\} = \phi$

c. [2 points] Let A be a set and C a partition of A. Then $\{\emptyset\} \notin C$.

False.

Let $A = \{\phi\}$. Then if $\mathcal{C} = \{\{\phi\}\}$,

we have $\{\phi\} \in \mathcal{C}$.

d. [2 points] The set $\mathbb{Z}_+ \times \mathbb{Z}_+$ with the dictionary order has the same order type as $\mathbb{Z}_+ \times \mathbb{Z}_-$ with the dictionary order, where $\mathbb{Z}_- = \{x \in \mathbb{Z} \mid x < 0\}$.

False.

$(1,1) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ is the smallest element.

However $\forall\ (a,b) \in \mathbb{Z}_+ \times \mathbb{Z}_-$,

$(a, b-1) < (a,b)$, so $\mathbb{Z}_+ \times \mathbb{Z}_-$ has no smallest element.