Sets

What is a set? Roughly, an unordered collection of distinct elements

"$x \in S$" means "x is an element of S"

x: element, S: set

A set is determined by its elements. That is, $A = B$ if and only if they contain the same elements.

Examples:

- $\mathbb{R}$ = real numbers (e.g. $\pi \in \mathbb{R}$)
- $\mathbb{Z}$ = integers = $\{\ldots,-2,-1,0,1,2,\ldots\}$ (e.g. $3 \in \mathbb{Z}$, but $1.5 \underset{\text{"is not in"}}{\notin} \mathbb{Z}$)
- U.S. states = {Alabama, Alaska, ...}
- $\{a,b,c\}$ (Note: $\{b,a,c\}$ is the same set — order doesn't matter! and $\{a,b,c\} = \{a,a,b,c,b\}$)
- $\mathbb{Q}$ = rational numbers = $\left\{\frac{a}{b} \,\middle|\, a,b \in \mathbb{Z},\ b \neq 0\right\}$ ("$|$" = "such that")

  = "numbers of the form $\frac{a}{b}$, such that a,b are integers, and $b \neq 0$"
- S = points on the unit circle = $\{(x,y) \in \mathbb{R}^2 \mid x^2+y^2=1\}$

  $\left((0,1) \in S,\ \left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right) \in S, \text{ but } (0,0) \notin S\right)$
- the empty set $= \varnothing = \{\,\}$ = the set with no elements = the set of integers that are both odd and even
- $\{1,7,\{5,10\}\}$ $= A$ (Note: $\{5,10\} \in S$, but $5 \notin S$)

Subsets

Def: A is a <u>subset</u> of B $(A \subseteq B)$ if every element of A is an element of B.

Note: $A=B$ if and only if $A\subseteq B$ and $B\subseteq A$.

Examples

- $\mathbb{Z}\subseteq\mathbb{Q}\subseteq\mathbb{R}\subseteq\mathbb{C}$ (complex #s)
- $\{b,a\}\subseteq\{a,b,c\}$
- If S is any set, $\emptyset\subseteq S$
- $\{5,10\}\not\subseteq\{1,7,\{5,10\}\}$, but $\underbrace{\{\{5,10\}\}}_{\text{(How many elements are in this set?)}}\subseteq\{1,7,\{5,10\}\}$
- $\emptyset\subseteq\{\emptyset\}$, but $\emptyset\neq\{\emptyset\}$ Why? $\{\emptyset\}$ has one element, $\emptyset$ has none
- $\underset{\text{set of pts on unit circle}}{S}\subseteq\mathbb{R}^2=\{(a,b)\mid a,b\in\mathbb{R}\}$
- "$\{a\}\subseteq A$" is equivalent to "$a\in A$."

Unions and Intersections

Def: Suppose A and B are sets.

a.) The union of A and B = $A\cup B$ = the set of elements that are in A or B (or both — but this is implied)

b.) The intersection of A and B = $A\cap B$ = the set of elements that are in both A and B

Note: $A \cap B \subseteq A \subseteq A \cup B$

$$\begin{array}{l} A \cap B \subseteq A \subseteq A \cup B \\ \rotatebox{90}{$\subseteq$} \\ B \subseteq A \cup B \end{array}$$

Examples:

- $\{1,2\} \cup \{1,3\} = \{1,2,3\}$, $\{1,2\} \cap \{1,3\} = \{1\}$
- $\{a,b,\{c,d\}\} \cap \{c,d\} = \emptyset$, $\{a,b,\{c,d\}\} \cup \{c,d\} = \{a,b,c,d,\{c,d\}\}$
- $\mathbb{Z} \cap \underbrace{\mathbb{R}_{>0}}_{= \{x \in \mathbb{R} \mid x > 0\}} = \mathbb{N} = \{1,2,3,\dots\}$
- $\{\emptyset\} \cup \emptyset = \{\emptyset\}$, $\{\emptyset\} \cap \emptyset = \emptyset$

Note: We can also define unions and intersections for collections (sets) of sets:

If $\mathcal{C}$ is a set of sets, then we define

$$\bigcup_{S \in \mathcal{C}} S = \{x \mid x \in S \text{ for some } S \in \mathcal{C}\}$$

$$\bigcap_{S \in \mathcal{C}} S = \{x \mid x \in S \text{ for all } S \in \mathcal{C}\}$$

Examples:

- For each $n \in \mathbb{N}$, define $S_n = \{1,2,\dots,n\}$, and $\mathcal{C} = \{S_1, S_2, S_3, \dots\}$

then $\bigcap_{n \in \mathbb{N}} S_n = \{1\}$ and $\bigcup_{n \in \mathbb{N}} \mathbb{N}$

- $\mathcal{C} = \{\{\emptyset\}, \{\{\emptyset\}\}, \dots\}$

  $\bigcap_{S\in\mathcal{C}} S = \emptyset$, $\bigcup_{S\in\mathcal{C}} S = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \dots\}$

- For each prime p, define $A_p = \{x \in \mathbb{N} \mid p \text{ divides } x\}$

  So $A_2 = \{2,4,6,\dots\}$, $A_3 = \{3,6,\dots\}$, etc. and $\mathcal{C} = \{A_p \mid p \text{ prime}\}$

  then $\bigcup_{A_p\in\mathcal{C}} A_p = \{2,3,4,\dots\}$ and $\bigcap A_p = \emptyset$

  Question: If $\mathcal{C}' \subseteq \mathcal{C}$ is a finite subset, what is $\bigcap_{A_p\in\mathcal{C}'} A_p$?

Basic Properties of $\cap$ and $\cup$:

- $A \cup \emptyset = A$, $A \cap \emptyset = \emptyset$
- $A \cup B = B \cup A$ (resp. $\cap$) (commutativity)
- $A \cup (B \cup C) = (A \cup B) \cup C$ (resp. $\cap$) (associativity)
- If $A \subset B$ then $A \cup B = B$ and $A \cap B = A$.
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$\left.\right\}$ (distributivity)

Proof of 1st distributive law: We want to show that the set on the left has the same elements as the set on the right.

If $x \in A \cap (B \cup C)$, then $x \in A$ and $x \in B$ or C. If $x \in B$ then

$x \in A \cap B$. If $x \in C$, then $x \in A \cap C$. Thus, $x \in (A \cap B) \cup (A \cap C)$.

If $x \in (A \cap B) \cup (A \cap C)$, then $x \in A$ and $x \in B$ or $x \in A$ and $x \in C$.

Thus, $x \in A$ <u>and</u> ($x \in B$ or $x \in C$), so $x \in A \cap (B \cup C)$.

[We will come back to proof strategies later!]